Sets

§1 Concepts and definition

1. Definition: A set is an unordered collection of elements.

2. Characteristics: (1) unequivocalness 确定性

   (2) unorderedness 无序性

   (3) distinctness 互异性

3. Notations:

   a belongs to A $\Leftrightarrow$ $a \in A$

   a doesn't belong to A $\Leftrightarrow$ $a \notin A$

4. special sets:

$\mathbb{R}$	$\mathbb{Z}$	$\mathbb{N}$	$\mathbb{Q}$	$\varnothing$
实数集	整数集	自然数集	有理数集	空集

5. special notations for intervals

- $[a,b] := \{x \in \mathbb{R} | a \leq x \leq b\}$
- $(a,b) := \{x \in \mathbb{R} | a < x < b\}$
- $[a,b) := \{x \in \mathbb{R} | a \leq x < b\}$
- $(a,b] := \{x \in \mathbb{R} | a < x \leq b\}$
- $[a,\infty) := \{x \in \mathbb{R} | x \geq a\}$
- $(a,\infty) := \{x \in \mathbb{R} | x > a\}$
- $(-\infty,b] := \{x \in \mathbb{R} | x \leq b\}$
- $(-\infty,b) := \{x \in \mathbb{R} | x < b\}$

6. subsets and supersets

set A is a subset of set B (or B is a superset of A) if every element of A is also in set B

$A \subseteq B$

☆ ϕ is a subset of any set

if $A \subseteq B$ and $A \neq B$, then A is a proper subset of B

$A \subset B$

7. 子集个数

set A 有 n 个元素 ($|A| = n$)

则 A 有 2^n 个子集，有 $2^n - 1$ 个真子集，有 $2^n - 2$ 个非空真子集

8. Power set

The power set of set A is the set of all subsets of A denoted as 2^A or $P(A)$

eg. $A = \{1, 2, 3\}$

$2^A = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

☆ $|2^A| = 2^{|A|}$

例题:

1. (quiz 例题) which is correct? D.

A. $\emptyset \in \{0\}$. B. $\{3\} \in \{1,3\}$ C. $0 \subseteq \{0,1\}$. D. $\emptyset \subseteq \{1\}$

$\in$. $\notin$ 表示元素与集合的关系. $\subseteq$ $\subset$ $\not\subset$ 表示集合与集合的关系

2. $A=\{x \mid -3 \le x \le 5\}$, $B=\{x \mid m-1 \le x \le 2m+1\}$, $B \subseteq A$. what is the set of all values for m?

① if $m-1 \le 2m+1 \Rightarrow m \ge -2$

then $\begin{cases} m-1 \ge -3 \\ 2m+1 \le 5 \end{cases} \Rightarrow -2 \le m \le 2$

$\therefore -2 \le m \le 2.$

② if $m-1 > 2m+1 \Rightarrow m < -2$

then $B = \emptyset$

$\because \emptyset \subseteq A$

$\therefore B \subseteq A$

In conclusion, $m \in (-\infty, 2]$

$\star$ $\emptyset$ is a subset of any set

3. (练习册原题) Suppose set $A = \{x \in R \mid ax^2+2x+1=0\}$ and there is at most one element in set A, then find the range of a.

① $a=0$

$2x+1=0 \Rightarrow x=-\frac{1}{2}.$

② $a \neq 0.$

$\Delta = 4-4a \le 0 \Rightarrow a \ge 1.$

In conclusion, $\{a \mid a=0 \text{ or } a \ge 1\}$

4. (练习册原题) Let set $\boxed{M=\{x \mid (x-a)(x^2-ax+a-1)=0\}}$ The sum of all elements in M is 3. Then find the value of a.

$x \quad -(a-1)$

$x \quad -1$

$(x-a)(x-a+1)(x-1)=0.$

$x_1=a, \quad x_2=a-1, \quad x_3=1$

① if $a \neq a-1 \neq 1$

then $a+a-1+1=3. \Rightarrow a=\frac{3}{2}.$

② if $a=1,$

then $a-1=0. \Rightarrow M=\{0,1\}.$ Sum$=1$ ✗

③ if $a-1=1.$

then $a=2 \Rightarrow M=\{1,2\},$ Sum$=3$ ✓

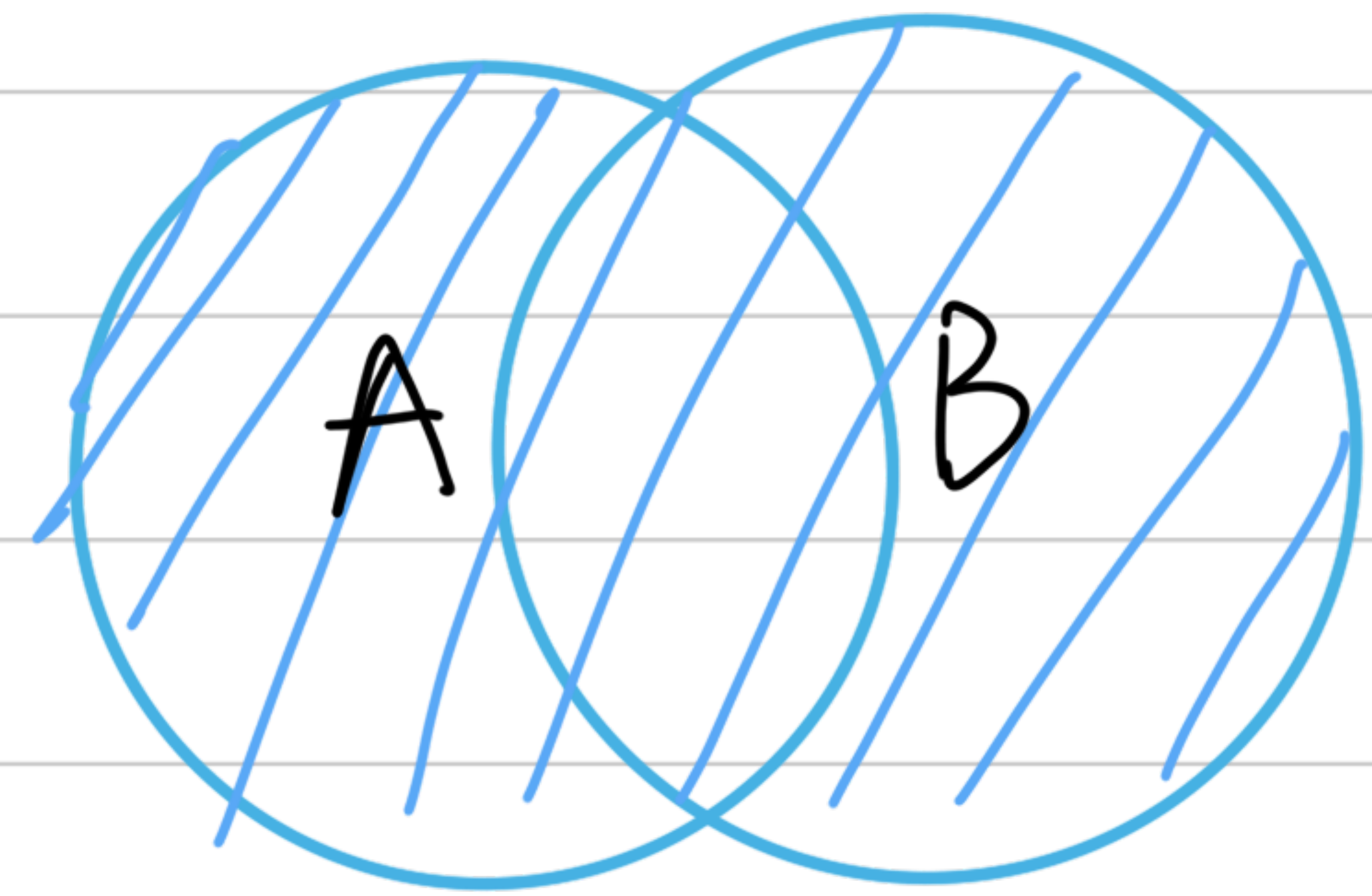
In conclusion, $a=\frac{3}{2}$ or $a=2$

§2 operations of sets

Union 并集

1. $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

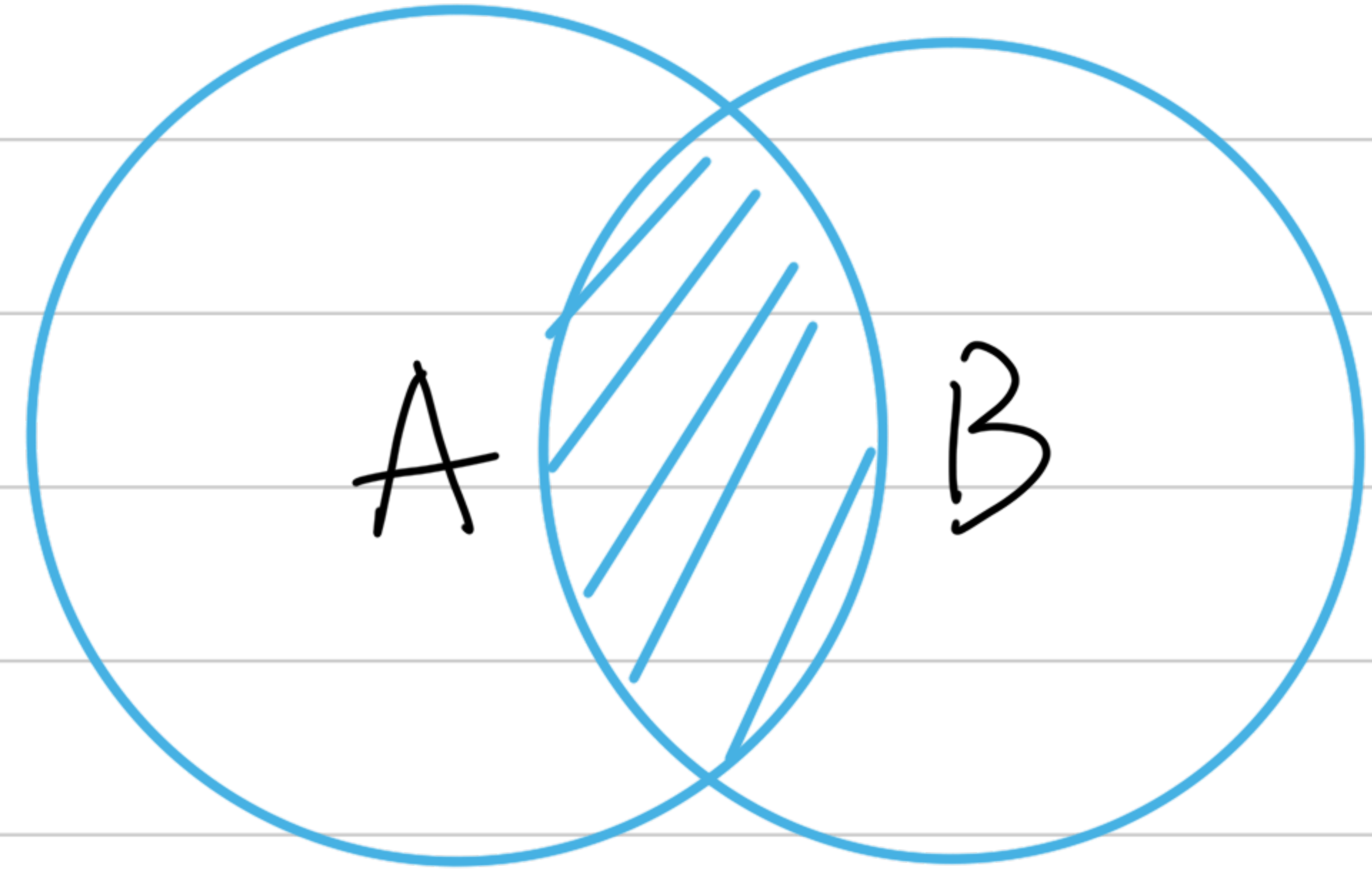
- $A \cup B = B \cup A$
- $A \cup (B \cup C) = (A \cup B) \cup C$
- $A \subseteq (A \cup B)$
- $A \cup A = A$
- $A \cup \varnothing = A$
- $A \subseteq B \Leftrightarrow A \cup B = B$



2. Intersections 交集

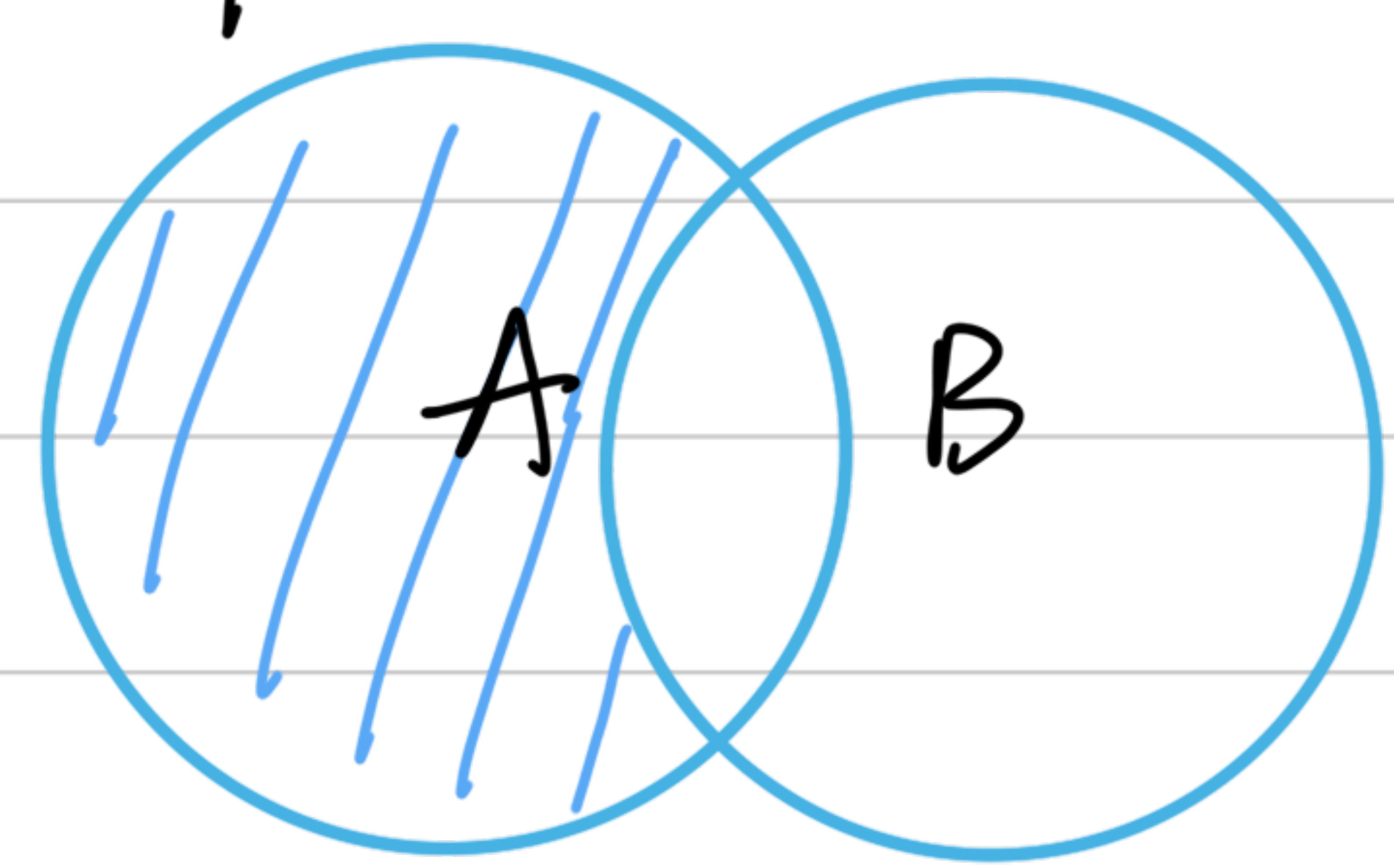
$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

- $A \cap B = B \cap A$
- $A \cap (B \cap C) = (A \cap B) \cap C$
- $(A \cap B) \subseteq A$
- $A \cap A = A$
- $A \cap \varnothing = \varnothing$
- $A \subseteq B \Leftrightarrow A \cap B = A$



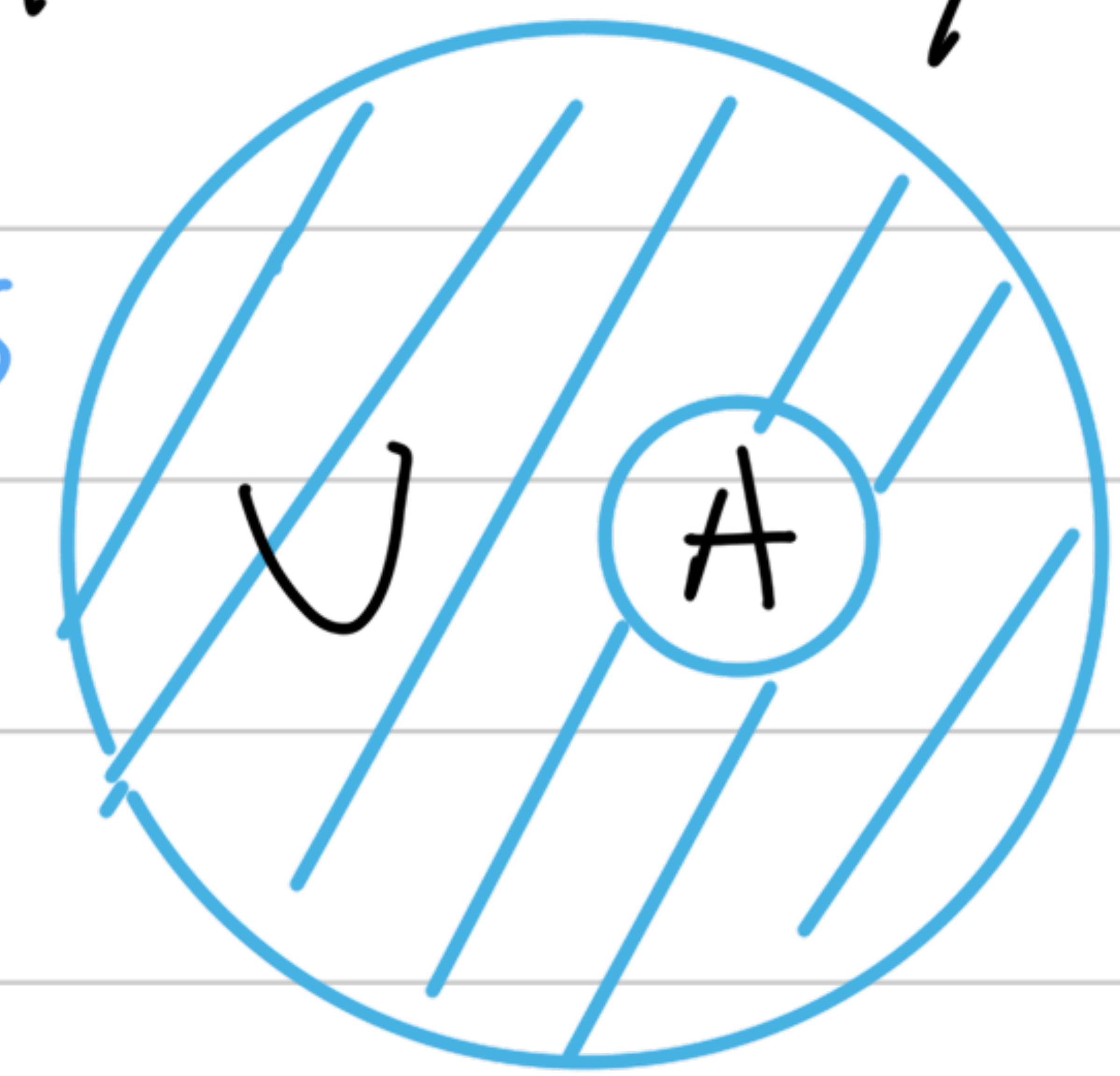
3. Complements 补集

$A \setminus B$ (or $A - B$) $= \{x \mid x \in A,\ x \notin B\}$



$U \setminus A$ ($\bar{A}$, or A^c) $= \{x \mid x \in U, x \notin A\}$

Universal set 全集 (a set that contains all the elements involved in the problem)



4. De Morgan's laws

$$(A \cup B)^c = A^c \cap B^c$$

$$(A \cap B)^c = A^c \cup B^c$$

5. Cartesian Product $A \times B$

$$A \times B = \{(x,y) \mid x \in A, y \in B\}$$

$$R^2 = R \times R = \{(x,y) \mid x \in R, y \in R\}$$

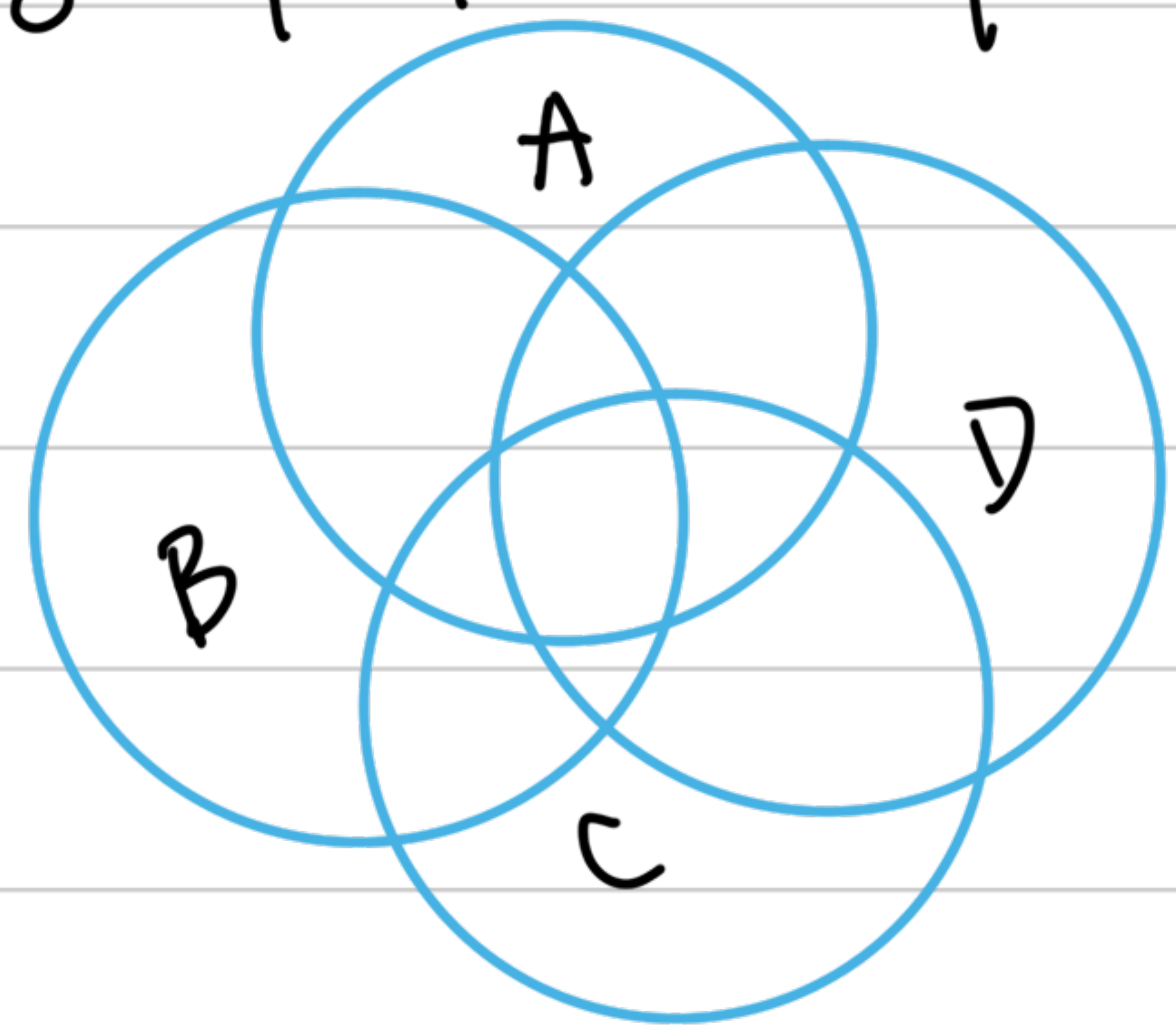
即  2-dimensional plane

6. Principle of inclusion and exclusion

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|$$



例题：1. $A=\{-1,2\}$, $B=\{x \mid ax-1=0\}$, $\boxed{A\cap B=B}$. What is the set of all values for a?

$B\subseteq A$.

① $\boxed{a\neq 0.}$

$ax-1=0 \Rightarrow x=\frac{1}{a}$.

$\frac{1}{a}=-1$ or $\frac{1}{a}=2$

$a=-1$ or $a=\frac{1}{2}$

② $\boxed{a=0}$

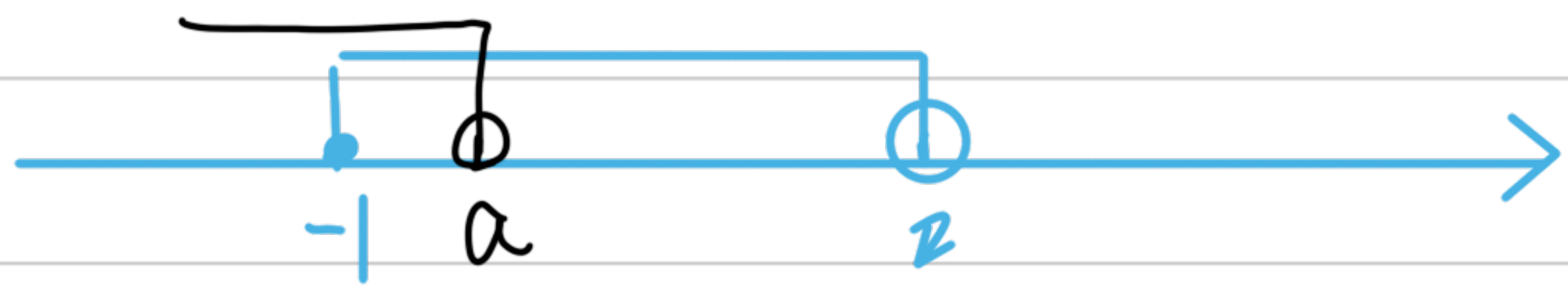
$-1=0 \Rightarrow B=\emptyset$

$\because \emptyset\subseteq A$

$\therefore B\subseteq A$

In conclusion, $a\in\{-1,\frac{1}{2},0\}$

2. $A=\{x \mid -1 \le x < 2\}$, $B=\{x \mid x<a\}$, $A \cap B \neq \emptyset$. What is the range of values for a?



$\therefore \{a \mid a > -1\}$

$a \neq -1$.
when $a=-1$
$A \cap B = \emptyset$

3. $\boxed{A=\{x \mid x^2-2x-8=0\}}$, $B=\{x \mid x^2+ax+a^2-12=0\}$, $\boxed{A \cup B \neq A}$. What is the set of all values for a?

$$\begin{matrix} x & x^2 \\ x & -4 \end{matrix}$$

$(x+2)(x-4)=0$
$x_1=-2,\ x_2=4$
$A=\{-2,4\}$

let's assume $A \cup B = A$.
then $B \subseteq A$
① $B \neq \emptyset$
if $-2 \in B$, then $4-2a+a^2-12=0$

$a_1=-2,\ a_2=4$

$a=-2,\ x^2-2x-8=0$ $\quad x_1=-2,\ x_2=4$ $\quad B=\{-2,4\}\subseteq A$

$a=4,\ x^2+4x+4=0$ $\quad x=-2$ $\quad B=\{-2\}\subseteq A$

if $4\in B$, then $16+4a+a^2-12=0$ $\quad a=-2$

Don't forget the empty set !!!

② $B=\varnothing$

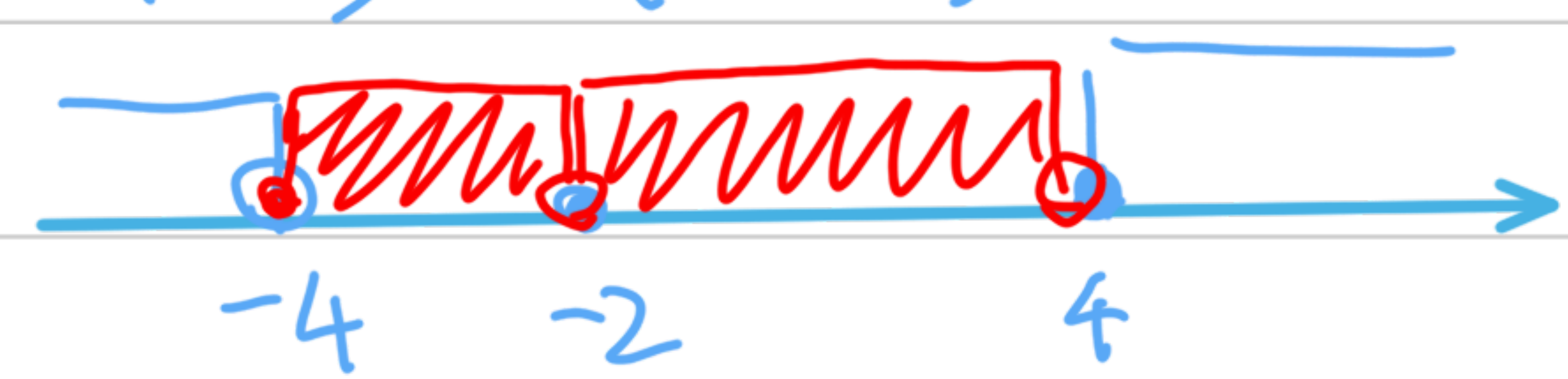
$\Delta = a^2-4(a^2-12)<0$

$a>4$ or $a<-4$

$\therefore\ a<-4$ or $a\geq 4$ or $a=-2$

when $A\cup B\neq A$, $-4\leq a<-2$ or $-2<a<4$

In conclusion, $a\in[-4,-2)\cup(-2,4)$



Logic

§1 statements

A statement is a sentence or a mathematical expression that is either true or false.

§2 quantifiers and negations

$\forall$ universal quantifier ("for every")

$\exists$ existential quantifier ("there exists")

The opposite of a statement p is called its negation $\neg p$

if a statement is true, its negation is false.

if a statement is false, its negation is true.

when writing the negation of a statement with quantifiers, we change $\forall$ to $\exists$ and change $\exists$ to $\forall$, and then negate the claim.

eg. p: $\forall x \in R, \exists y \in R, x+y=5$

$\neg p$: $\exists x \in R, \forall y \in R, x+y \neq 5$

§3 Compound statements and truth table

$p \wedge q$ is true when both p and q are true

$p \vee q$ is true when at least one of p or q is true

p	q	$\neg p \wedge q$	$p \wedge \neg q$	$(\neg p \wedge q) \vee (p \wedge \neg q)$	$(\neg p \vee \neg q) \wedge (p \vee q)$
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	F	F	F	F

§4 conditional statements

$P \Rightarrow Q$

P is a sufficient condition for Q

Q is a necessary condition for P

1. Converse and Contrapositive

$P \Rightarrow Q$

the converse: $Q \Rightarrow P$

the contrapositive $\neg Q \Rightarrow \neg P$

if $P \Rightarrow Q$, it is not necessarily true that $Q \Rightarrow P$

But if $P \Rightarrow Q$, $\neg Q \Rightarrow \neg P$ is always true.

eg. if x^2-6x+5 is even, then x is odd.

Converse: if x is odd, then x^2-6x+5 is even

Contrapositive: if x is even, then x^2-6x+5 is odd.

例题: 1. $\boxed{A=\{x \mid x^2-8x-20\le 0\}}$, $\boxed{B=\{x \mid 1-m\le x\le 1+m\}\ (B\neq\emptyset)}$, $x\in A$ is a necessary but not sufficient for $x\in B$. What is the set of all values for m?

$x\in B \Rightarrow x\in A$

$\Rightarrow B\subset A$. (proper s

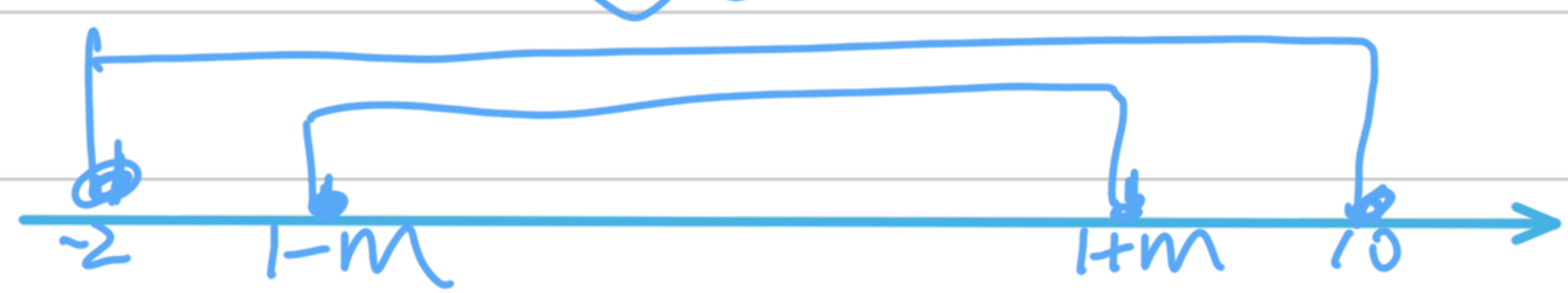
$\because B\neq\emptyset$

$\therefore 1-m\le 1+m$

$\underline{m\ge 0}$.

$(x+2)(x-10)\le 0$

$-2\le x\le 10$.



$$\begin{cases} -2 \le 1-m \\ 10 \ge 1+m \end{cases}$$

$$\begin{cases} m \le 3 \\ m \le 9 \end{cases}$$

$\therefore m \le 3$

$\therefore m \ge 0$

$\therefore 0 \le m \le 3$

when $1-m=-2$, $m=3$

$1+m=4 \ne 10$

when $1+m=10$, $m=9$

$1-m=-8 \ne -2$

$\therefore B \ne A$

可以取等

In conclusion, $m \in [0, 3]$

2. "$\forall x \in R, ax^2-2ax+3>0$" is false. What is the set of all values for a?

Method 1: $\exists x \in R, ax^2-2ax+3 \leq 0$ is true.

① $a=0$ $\quad 3 \leq 0$ × (left out)

② $a<0$. always true.

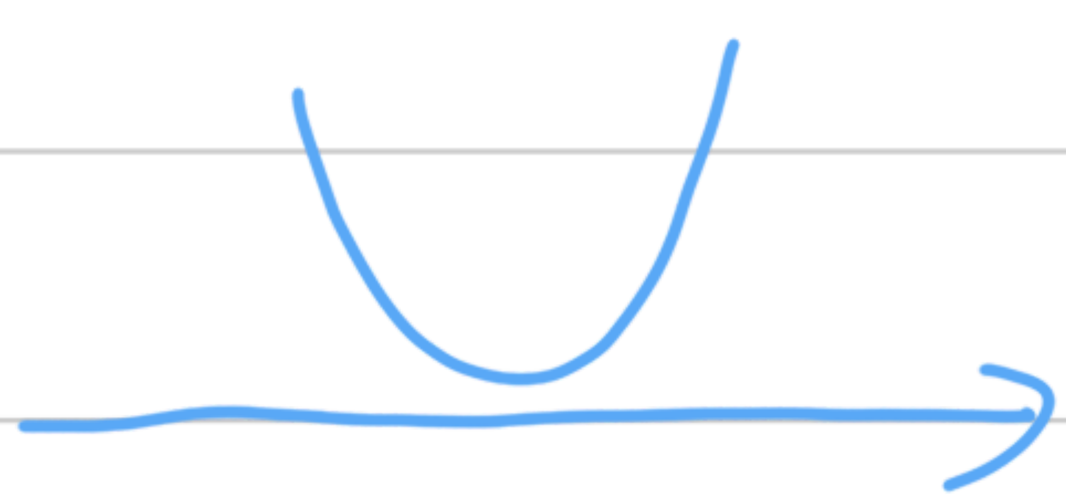
③ $a>0$. $\Delta = 4a^2-12a \geq 0$

$a \leq 0$ or $a \geq 3$

$\therefore a \geq 3$

In conclusion, $a \in (-\infty, 0) \cup [3, +\infty)$

Method 2: Let's assume "$\forall x \in R, ax^2-2ax+3>0$" is true.



① $a \neq 0$.

$$\begin{cases} a>0 \\ \Delta = 4a^2-12a<0. \end{cases}$$

$\begin{cases} a>0 \\ 0<a<3 \end{cases}$

$\therefore 0<a<3$

② $a=0$

$3>0$ always true

$\therefore 0 \le a<3$

In conclusion, $a \in (-\infty, 0) \cup [3, +\infty)$

Functions

1. Definition

$f: A \to B$ A对应B

domain 定义域　　range 值域

If function have the same domain A

$a \in A$, $f(a) = g(a)$

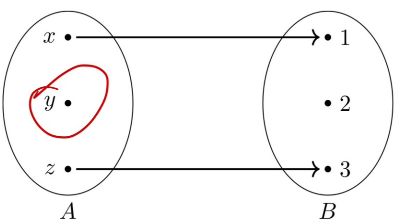


Figure 4.3: **NOT** a function/mapping. y is not mapped.

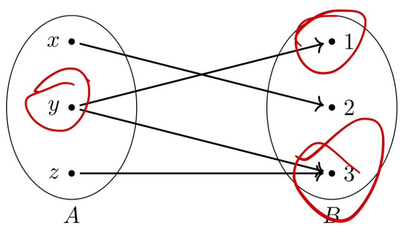
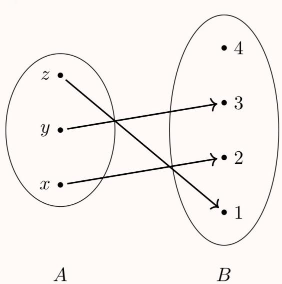


Figure 4.4: **NOT** a function/mapping. y is mapped to two elements in B

2. Injective 单射 (one-to-one)



re 4.5: An injection from $\{x, y, z\}$ to $\{1, 2, 3, 4\}$

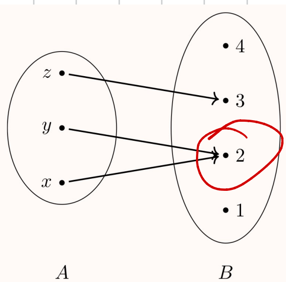


Figure 4.6: **Not an injection** from $\{x, y, z\}$ to $\{1, 2, 3, 4\}$

if $a_1 \neq a_2$

$\Downarrow$

$f(a_1) \neq f(a_2)$

3. surjective (or onto) 满射

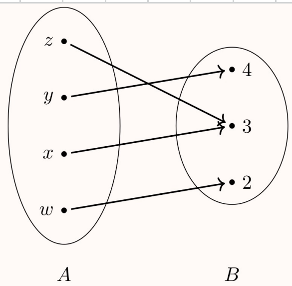


Figure 4.7: A surjection from $\{w, x, y, z\}$ to $\{2, 3, 4\}$

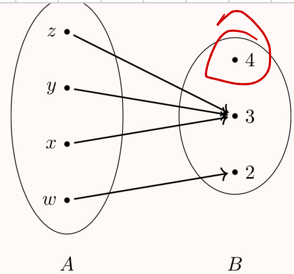


Figure 4.8: **Not** a surjection from $\{w, x, y, z\}$ to $\{2, 3, 4\}$

$f: A \to B$

one-to-on (injective) $|A| \leq |B|$

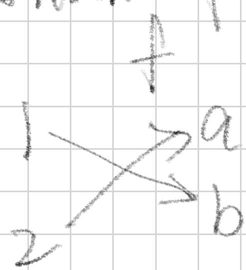
on-to (surjective) $|A| \geq |B|$

4. Function Composition

$f \circ g(x) = f(g(x))$

compose结合 先算后面的函数.

5. Inverse Function 反函数



$f^{-1}(a) = 2$

$f^{-1}(b) = 1$

$f: A \to B$ $\quad f^{-1}: B \to A$

$f(x)$ 与 $f^{-1}(x)$ 关于 $y=x$ 对称

a function is invertible on a domain if and only if it is injective

6. domain 定义域

$f(x)=\sqrt{x}$ $\{x \mid x \geq 0\}$

$f(x)=\frac{1}{x}$ $\{x \mid x \neq 0\}$

$f(x)=x^0$ $\{x \mid x \neq 0\}$

7. range 值域

$f(x)=\frac{ax+b}{cx+d}$ domain $=(-\infty,-\frac{d}{c})\cup(-\frac{d}{c},+\infty)$

If $y=\frac{ax+b}{cx+d}$

$cyx+dy=ax+b$

$dy-b=(a-cy)x$

$x=\frac{dy-b}{a-cy}$

$a-cy\neq 0$

$y\neq\frac{a}{c}$ $\therefore$ range $=(-\infty,\frac{a}{c})\cup(\frac{a}{c},+\infty)$

8. Graphs of Functions

Shifting 平移

$f'(x)=f(x-a)+b$ → 上加下减

左加右减

9. Reflection Across a Horizontal Line 关于水平线对称.

If reflection on $y=b$

$f'(x)=2b-f(x)$

10. Reflection Across a Vertical Line 关于竖线对称.

If reflection on $x=a$

$f'(x)=f(2a-x)$

11. Reflection Across a point

If point (a,b)

$f'(x)=2b-f(2a-x)$

12. If $f(x)=f(2a-x)$ 则说明 $x=a$ 是此函数对称轴.

e.g. 证2次函数对称轴

$f(x)=ax^2+bx+c$

$f(\cancel{2}\cdot\frac{-b}{\cancel{2}a}-x)=f(\frac{-b}{a}-x)$

$=a(\frac{-b}{a}-x)^2+b(\frac{-b}{a}-x)+c$

$=a(\frac{b^2}{a^2}+\frac{2bx}{a}+x^2)-\frac{b^2}{a}-bx+c$

$=\cancel{\frac{b^2}{a}}+2bx+ax^2-\cancel{\frac{b^2}{a}}-bx+c$

$=ax^2+bx+c=f(x)$

13. Even and Odd Functions.

- Even functions
  $f(-x)=f(x)$ 关于y轴对称
- Odd Functions
  $f(-x)=-f(x)$ 关于原点对称

※ 函数的domain必须关于原点对称 才可判断 even or odd.

- 运算

f	g	$g+f$	$f-g$	$g\cdot f$	$\frac{f}{g}$
even	even	even	even	even	even
odd	odd	odd	odd	odd	odd.

f	g	$f\circ g$
odd	odd	odd
even	odd	even
any	even	even

14、Monotonicity 单调性

if $\forall x, y \in D$, $x>y \Rightarrow f(x)>f(y)$ increasing

$x>y \Rightarrow f(x)<f(y)$ decreasing.

f	g	$f\circ g$

↑	↑	↗
↗	↘	↘
↘	↗	↘
↘	↘	↗

函数例题

1. 求 Inverse Functions

① (b) $f(x) = \frac{1+3x}{5-2x}$

$$y = \frac{1+3x}{5-2x}$$

$$5y - 2yx = 1 + 3x$$

$$5y - 1 = (3+2y)x$$

$$x = \frac{5y-1}{3+2y}$$

$$f^{-1}(x) = \frac{5x-1}{3+2x}$$

(f) $f(x) = 1 - x^3$

$$y = 1 - x^3$$

$$x^3 = 1 - y$$

$$x = \sqrt[3]{1-y}$$

$$f^{-1}(x) = \sqrt[3]{1-x}$$

解反函数的方法与求range的方法相似

② find the inverse function of $f(x) = x^2 + 1$

$$f(x) = x^2 + 1$$

$$y = x^2 + 1$$

No inverse function

$x^2 = y-1$

$x = \pm\sqrt{y-1}$

x有两个解（f(x)必须是one-by-one才能有inverse function）

2、求 domain

(a) $f(x) = \frac{\sqrt{x}}{2x^2+x-1}$

$\sqrt{x} \geq 0$

$x \geq 0$

$2x^2+x-1 \neq 0$

$(2x-1)(x+1) \neq 0$

$x \neq \frac{1}{2}$ or -1

$\because x \geq 0$

$\therefore \{x \mid x \geq 0 \text{ and } x \neq \frac{1}{2}\}$

(d) $f(x) = \frac{1}{1+\frac{1}{1+\frac{1}{x}}}$

$x \neq 0$

$1+\frac{1}{x} \neq 0$

$x \neq -1$

$1+\frac{1}{1+\frac{1}{x}} \neq 0$

$\frac{1}{1+\frac{1}{x}} \neq -1$

$x \neq -\frac{1}{2}$

$\{x \mid x \neq 0, x \neq -1, x \neq -\frac{1}{2}\}$

3、求 range

Find the range of $f(x) = \frac{1-x}{1+x}$

F_1: $f(x) = \frac{1-x^2}{1+x^2}$

$y + yx^2 = 1 - x^2$

$(y+1)x^2 = 1-y$

$x^2 = \frac{1-y}{y+1} \geq 0$

$(1-y)(y+1) \geq 0$

$[-1, 1]$ ∵ $y+1 \neq 0$

$y \neq -1$

∴ $(-1, 1]$

F_2: $f(x) = \frac{1-x^2}{1+x^2}$

$= -\frac{x^2-1}{x^2+1}$

$= -\frac{x^2+1-2}{x^2+1}$

$= -1 + \frac{2}{x^2+1}$

$x^2+1 \geq 1$

$0 < \frac{2}{x^2+1} \leq 2$

$-1 < -1 + \frac{2}{x^2+1} \leq 1$

$(-1, 1]$

4. Prove Odd and Even functions

(1) $f(x) = x^2 + \frac{1}{|x|}$

$f(-x) = (-x)^2 + \frac{1}{|-x|}$ even

$= x^2 + \frac{1}{x} = f(x)$

(2) $f(x) = |x+1| - |x-1|$

$f(-x) = |-x+1| - |-x-1|$

$= |-(x-1)| - |-(x+1)|$

$= |x-1| - |x+1| = -f(x) = -|x+1| + |x-1|$

odd

5. Discuss the monotonicity

4. Discuss the monotonicity of $f(x) = \frac{2x+1}{x-2}$ and prove your statement.

when $x < y < 2$ $\quad f(x) - f(y) = \frac{2x+1}{x-2} - \frac{2y+1}{y-2}$

$\because y > x$

$y - x > 0$

$= \frac{-5x+5y}{(x-2)(y-2)}$

$x < 2 \quad y < 2$

$= \frac{5(y-x)}{(x-2)(y-2)}$

$x-2 < 0 \quad y-2 < 0$

$(x-2)(y-2) > 0$

$\therefore \frac{5(y-x)}{(x-2)(y-2)} > 0$ decreasing

when $x > y > 2$ $\quad f(x) - f(y) = \frac{\overset{<0}{5(y-x)}}{\underset{>0}{(x-2)}\underset{>0}{(y-2)}} < 0$

decreasing

6. Simple Functional Equations

7. Given $f(3x + 1) = 4x + 7$, then what is $f(x)$?

① if $3x+1 = t$ ② $f(3x+1) = \frac{4}{3}(3x+1) + 7 - \frac{4}{3}$

$$x = \frac{t-1}{3}$$

$$f(t) = \frac{4t-4}{3} + 7$$

$$= \frac{4t}{3} - \frac{17}{3}$$

$$f(x) = \frac{4x}{3} - \frac{17}{3}$$

$$f(x) = \frac{4}{3}x + 7 - \frac{4}{3}$$

$$= \frac{4}{3}x - \frac{17}{3}$$