SETS

1.1 Concepts and definition

1. Concepts of set : a list or collection of element that share a characteristic

2. Three characteristics:

① Unequivocalness（确定性）

② Unorderedness（无序性） {1, 2} = {2, 1}

③ Distinctness（不重复性） {1, 1} should be {1}

3. Notations

① While element " a belongs to A " , we represent it as $a \in A$

② While element " a doesn't belong to A " , we represent it as $a \notin A$

4. Finite 有限集 {1,2,3}

   Infinite 无限极 {1,2,3...}

5. Special sets

① N:set of natural numbers（自然数）

② Z:set of all integers（整数）

③ Q:set of all rational numbers（有理数）

④ R:set of all real numbers（实数）

6. (→ 不取等 [→ 取等 ∞ → 不取等

7. Cardinality means the numbers of elements of the set

$A=\{1,2,3\}$ $n(A)=3$ $B=\{a,b,c,\{1,2\}\} \Longrightarrow n(B)=4, |B|=4$

8. If every element of A is also in set B, then A is B's subset（子集）,we represent it as $A \subseteq B$

9. If $A \subseteq B$ and $A \neq B$, then A is a proper subset（真子集）of B, $A \subset B$

$A=\{1,2,3\}$ $B=\{1,2,3,4\}$ $A \subseteq B$ and $A \subset B$

Proper set must be involved in subset. If A is the proper subset of B, then A must be the subset of B.

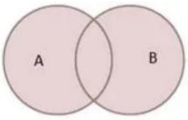
10. The power set of set A is set of all subsets of A, we represent is as 2^A

$A=\{1,2,3\}$

$2^A=\{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

1.2 Operations of sets

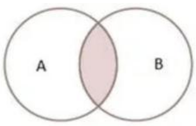
1. Unions 并集 $A \cup B=\{x|x \in A \text{ or } x \in B\}$



$\{1,2\} \cup \{2,3\} = \{1,2,3\}$

The pink area in the picture above represents the union of A and B.

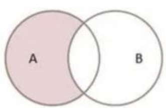
2. Intersections 交集 $A \cap B=\{x|x \in A,\ x \notin B\}$



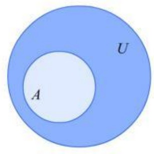
$\{1,2\} \cap \{2,3\} = \{2\}$

The pink area in the picture above represents the intersections of A and B.

3. Complements 补集 $A \setminus B(A-B)=\{x|x \in A, x \notin B\}$



The pink area in the picture above represents the complements of A and B.



$C_U A$ A^U / $U\backslash A$

4. De Morgan's laws: The complement and unions and intersections together satisfy an interesting property.

$(A\cup B)^C = A^C \cap B^C$ $(A\cap B)^C = A^C \cup B^C$

5. Catesian product $A\times B=\{(x,y)\mid x\in A, y\in B\}$

Suppose $A=\{1,2\}$, $B\ \{a,b\}$, then $A\times B=\{(1,a),(1,b),(2,a),(2,b)\}$.

(the first element is from A and the second is from B)

Numbers of elements: $n(A)\times n(B)=2\times 2=4$ = numbers of elements of Cartesian product

6. 容斥原理

$$\left|\bigcup_{i=1}^{n} A_i\right| = \underbrace{\sum_{i=1}^{n} |A_i|}_{\text{sum of each set}} - \underbrace{\sum_{1\le i<j\le n} |A_i \cap A_j|}_{\text{sum of each 2-intersection}} + \underbrace{\sum_{1\le i<j<k\le n} |A_i \cap A_j \cap A_k|}_{\text{sum of each 3-intersection}} - \ldots + (-1)^{n+1}|A_1 \cap A_2 \cap \ldots \cap A_n|$$

Exercise for SETS

1. Given that A is a set of numbers such that : if $a\in A$, then $\frac{1}{1-a}\in A$ ($a\neq 1$). Soppose we know $2\in A$. Find the set A with the smallest number of elements.

$$\because 2\in A \quad \therefore -1\in A$$

$$\because -1\in A \quad \therefore \frac{1}{2}\in A$$

$$\because A=\{-1,2,\frac{1}{2}\}$$

2. Let set $M=\{x|(x-a)(x^2-ax+a-1)=0\}$. The sum of elements in M is 3. Then find the value of a.

$$(x-a)[(x+1)(x-1)-a(x-1)]=0$$

$$(x-a)[(x+1-a)(x-1)]=0$$

$$(x-a)(x+1-a)(x-1)=0$$

$$So\ \ x_1=a, x_2=a-1, x_3=1$$

① $a+a-1+1=3$

$2a=3$

$a=\frac{3}{2}$

② $x_2=x_3=a-1=a$

$a+1=3$

$a=2$

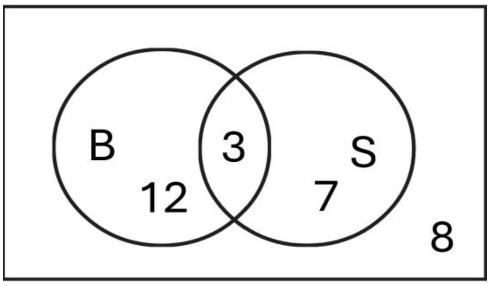
$$In\ all,\ a=\frac{3}{2}/2$$

3. Given A {a,b}, list all elements of $A\times 2^A$ (2^A means power point set, x means Cartesian product)

$$2A=\varnothing\ ,\ \{a\}\ ,\ \{b\}\ ,\ \{a,b\}$$

$A \times 2A =$ { {a,∅ }, {a {a} }, {a {b} }, {a {a,b} }, {b,∅ }, {b {a} }, {b {b} }, {b {a,b} } }

4. There are 30 students in a class. 15 of them like basketball, 10 of them like soccer, 8 of them like neither. What is the number of students who like basketball but don't like soccer



$30-8-10=12$

5. Let sets $A=\{x|x\leqslant 6, x\in Z^+$, $B=\{x|x$ is not prime number$\}$ and $C=A\cap B$, then there are ___ non-empty subset of C.

$x\in\{1,2,3,4,5,6\}$ prime number under 7 include 2,3,5

$\therefore C=$ {∅, {2}, {3}, {5}, {2,3}, {3,5}, {2,5}, {2,3,5} }

So the answer is 6

6. Find $\{x-y|-1\leqslant x\leqslant y\leqslant 1\}$ =___

$\because x\leqslant y \quad \therefore MAX_{x-y}=0 \qquad MIN_{x-y}=(-1)-1=-2$

So the answer is $[-2,0]$

LOGIC

2.1 Statements（命题）

1.A statement is a sentence or a mathematical expression that is either true or false

2.①Every theorem（定理）/proposition（命题）/lemma（引理）/corollary（推论） is a statement and true

②Every conjecture is a statement (of unknown truth value)

③Every incorrect calculation or deduction is a false statement

2.2 Quantifiers and negations

1.Quantifiers（量词）

①$\forall$ means "for all" or "for every" or "for each",called as universal quantifier

②$\exists$ means "there exists" or "for some",called as a existential quantifier

③when both quantifiers appear in a statement

2.negations（否定）

①when investivating a statement is true, we some times went to approach it in the opposite way, $\neg p$ is negation

②if the statement is V, then the negation is F

if the statement is F, then the negation is T

③when writing the negations, change $\exists$ to $\forall$ or change $\forall$ to $\exists$, and negate the chaim, not domain of our concern!

3.Compound statements and truth table

①if p and q is true, then $p \wedge q$ is true

②if one of p&q is true, then $p \vee q$ is true

p	q	$\neg p \wedge q$	$p \wedge \neg q$	$A \vee B$	$p \vee q$	$p \wedge q$	$\neg p \vee \neg q$
T	T	F	F	F	T	T	F
T	F	F	T	T	T	F	T
F	T	T	F	T	T	F	T
F	F	F	F	F	F	F	T

2.4 Conditional statements

1. "P implies Q" is denoted as $P \Rightarrow Q$.

"P if and only if Q" is denoted as $P \Leftrightarrow Q$.

2. Sufficient condition: If there is a condition A, there must be a result B, then A is a sufficient condition for B. That is, $A \Rightarrow B$.

3. Necessary condition: if there is no condition A, there must be no result B, then A is the necessary condition of B. That is, $a \Rightarrow b$ is equivalent to $b \Rightarrow a$.

4. Relationship

①Sufficient conditions are not necessarily necessary conditions, and necessary conditions are not necessarily sufficient conditions.

②If A is a sufficient condition for B, then B is a necessary condition for A.

③If a is a necessary condition for b, then b is a sufficient condition for a.

④A is a necessary and sufficient condition for B, if and only if A is a sufficient condition for B and A is a necessary condition for B.

Exercise for LOGIC

1.Student A indicates that B is lying, student B indicates that C is lying. However, C says that both A and B are lying. Who tells the truth?

	T	L
A		√
B	√	
C		√

If A is true, then B is lying, so C is telling the truth. While C is telling the truth, then A and B are both lying. It doesn' t make sense. So A is lying.

If A is lying, then B is telling the tuth, so C is lying. As a result, A and B aren' t both lying, which make sense.

2. "Proposition $p \wedge q$ is false（A）" is the ____ condition for "Proposition $p \vee q$ is false（B）".

(A) Sufficient but not necessary

(B) Necessary but not sufficient

(C) Both sufficient and necessary

(D) Neither sufficient nor necessary

For proposition p and q, Proposition $p \wedge q$ is false means p and q are false or one of them is true and the another one is false. Proposition $p \vee q$ is false means p and q are false. So A is B' s not sufficient but necessary. Because if proposition $p \wedge q$ means one of them is true and the another one is false, then we can' t infer

Proposition $p \vee q$ is false. But if Proposition $p \vee q$ is false, we can infer Proposition $p \wedge q$ is false. So the answer is B.

3.Write the converse of "If $a+c>b+d$, then $a>b$, $c>d$"

If $a>b$, $c>d$, then $a+c>b+d$.

4.Write the contrapositive of "If $x>1$, then $(x+1)(x-1)>0$"

If $(x+1)(x-1)<0$, then $x \leqslant 1$

5.Write the negation of "$\forall x \in R, \exists y>0$, $y=x^2$"

$\exists y>0, \forall x \in R, y \neq x^2$

6.Given two statements p、q, consider two expressions $E_1=(\neg p \wedge q) \vee (p \wedge \neg q)$, $E_2=(\neg p \vee \neg q) \wedge (p \vee q)$. Construct truth tables for these two expressions based on whether p, q are true or false, and prove these two expressions are equivalent.

p	q	$\neg p \wedge q$	$p \vee \neg q$	$\neg p \vee \neg q$	$p \vee q$	E_1	E_2
T	T	F	F	F	T	F	F
T	F	F	T	T	T	T	T
F	T	T	F	T	T	T	T
F	F	F	F	T	F	F	F

SOLVING INEQUALITIES

3.1 Properties of inequalities

1.Transitive Property

If $a<b$ and $b<c$, then $a<c$. In notations: $(a<b)\wedge(b<c)\Rightarrow(a<c)$.

2.Addition of Inequalities:

If $a<b$ and $c<d$, then $a+c<b+d$. In notations: $(a<b)\wedge(c<d)\Rightarrow a+c<b+d$.

3.Addition of a Constant:

If $a<b$, then $a+c<b+c, \forall c\in R$. In notations: $(a<b)\ \ =\Rightarrow a+c<b+c, \forall c\in R$.

4.Multiplication by a Constant:

If $c>0$, $a<b$, then $ac<bc$. If $c<0$, $a<b$, then $ac>bc$.

In notations: $(a<b)\wedge(c>0)\Rightarrow ac<bc$. $(a<b)\wedge(c<0)\Rightarrow ac>bc$.

3.2Absolute Value Inequalities

When solving inequalities (or equations)with absolute values, we need to separate the real line into disjoint intervals so that on each set, the absolute value can be simplified according to its sign: $x>0\Rightarrow|x|=x, x\leqslant 0\Rightarrow|x|=-x$. Inside each case, the solution is obtained by intersecting that set with the solution of the inequality; overall the solution set is the union of the solution set of all different cases.

Solve $|x+2|+|2x+5|<10$.

$$x+2=0\Rightarrow x=2,\ \ 2x-5=0\Rightarrow x=\frac{5}{2}$$

Thus $-2,\frac{5}{2}$ divides the real line into three disjoint parts.

① if $x \leqslant -2$, then $|x+2| = -x-2$, $|2x-5| = 5-2x$

$$-x-2+5-2x<10$$

$$x>-\frac{7}{3}$$

In this case the solution is set $(-\frac{7}{3},-2]$

② if $-2<x\leqslant\frac{5}{2}$, then $|x+2|=x+2$, $|2x-5|=5-2x$

$$X+2+5-2x<10$$

$$x>-3$$

Thus in this case the solution set is $(-3,\infty) \cap (-2, \frac{5}{2}] = (-2, \frac{5}{2}]$

3.3Quadratic Inequalities

1.If a quadratic equation $ax^2+bx+c=0$ has two non-equal real solutions α, β, then it can be factored into $ax^2+bx+c=a(x-a)(x-\beta)$

2.Without loss of generality, we can assume $a>0$, $a<\beta$. If $a<0$, we can always transform the inequality into the case where $a>0$. Then by inspecting the sign of $(x-a)(x-\beta)$, we can compile the following table.

x	$x<a$	$x=a$	$a<x<\beta$	$x=\beta$	$x>\beta$
$x-a$	-	0	+	+	+
$x-\beta$	-	-	-	0	+
$(x-a)(x-\beta)$	+	0	-	0	+

3.4 Polynomial Inequalities

$P(x)=a_n x_n + a_{n-1} x_{n-1} + \cdots\cdots + a_1 x + a_0 = 0$

$P(x)=a(x-x_1)(x-x_2)\ldots(x-x_n)$

In this case $a(x-x_1)(x-x_2)\ldots(x-x_n)=0$ has n solutions x_1, x_2, x_3,...., x_n, and they divide the real line into intervals, we just need to identify the sign of the polynomial expression on each interval.

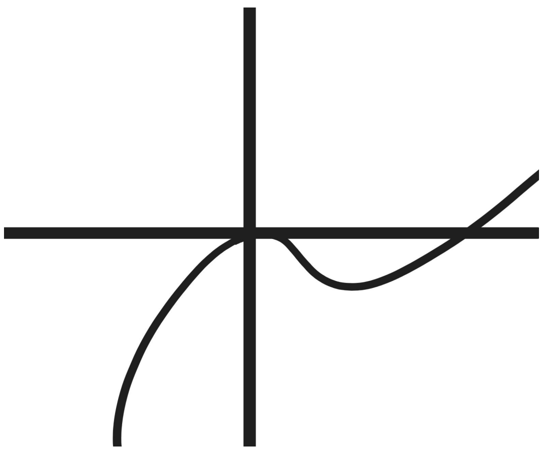
3.5 Rational Inequalities

穿针引线：奇穿偶不穿，奇变偶不变，符号看象限

已知方程 $x^3-x^2-k=0$ 有三个不同的实数根，则 k 的取值范围是 ____

取辅助函数 $f(x)=x^3-x^2=x^2(x-1)$

$x=0$ 为偶次零点，$x=1$ 为奇次零点（如下图）



$f'(x)=3x2-2x=0 \Rightarrow$ 极值点为 $x=\frac{2}{3}, x=0$

极值为 $f(\frac{2}{3})=(\frac{2}{3})^2(\frac{2}{3}-1)=-\frac{4}{27}$，$f(0)=0$

故 k 的取值范围为 $(-\frac{4}{27}, 0)$

Exercise for solving inequalities

1. $|x+1|+|x-3|<6$

When $x+1=0$, then $x=-1$. When $x-3=0$, then $x=3$.

① $x\geqslant 3$ $\quad x+1+x-3<6$

$2x-2<6$

$2x<8$

$x<4$

So $3\leqslant x<4$

② $-1\leqslant x<3$

$x+1-x+3<6$

$4<6$ always make sense

So $-1\leqslant x<3$

③ $x<-1$ $\quad -x-1-x+3<6$

$-2x+2<6$

$2x>-4$

$x>-2$

So $-2<x<-1$

In all, $-2<x<4$

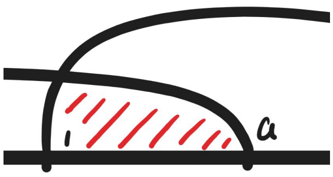
2. $6x^2-13x+5<0$

$(3x-5)(2x-1)<0$

So $\frac{1}{2}<x<\frac{5}{3}$

3. Solve the inequality for x: $x^2-(a+1)x+a<0$

$(x-a)(x-1)<0$



So $(1,a)\cup(a,1)$

4. $\frac{2x^2+1}{x-2} \leqslant -1$

①if $x-2=0$

Then it doesn't make sense

②if $x-2>0$

Then $2x^2+1 \leq 2-x$

$2x^2+x-1 \leq 0$

$(x+1)(2x-1) \leq 0$

$-1 \leq x \leq \frac{1}{2}$

Don't make sense

③if $x-2<0$

$2x^2+1 \geq 2-x$

$2x^2+x-1 \geq 0$

$(x+1)(2x-1) \geq 0$

$x \geq \frac{1}{2}$ or $x \leq 1$

In all, $\frac{1}{2} \leqslant x < 2$ or $x \leqslant -1$

GENERAL FUNCTIONS

4.1 Mappings Functions $f(x)$

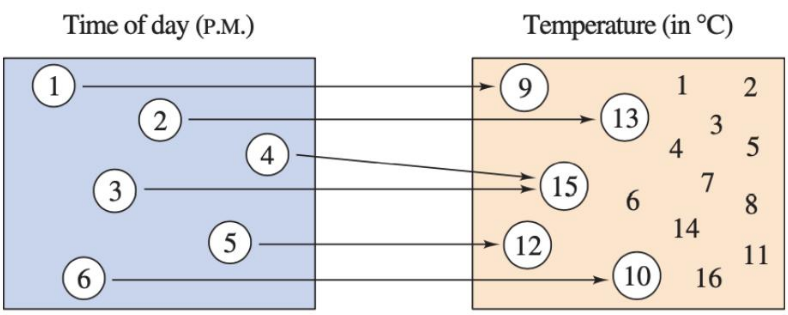
1.Domain（定义域）： A domain is a set of all possible values of an independent variable (usually expressed by x) in a function.

The domain of function $y=\sqrt{x}$ is $x \geqslant 0$

2.Range（值域）： A range is a set of all possible values of a function dependent variable (usually represented by y).

The range of function $y=\sqrt{x}$ is $y \geqslant 0$

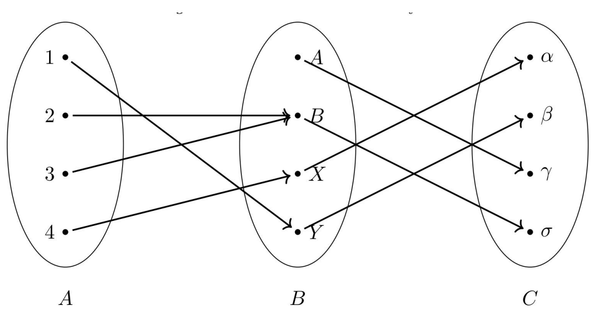
3.Also note that it is perfectly legit that more than one elements in A are mapped to the same element in B, or some elements in B are not mapped on by any element in A.



Set A is the domain.
Inputs: 1, 2, 3, 4, 5, 6

Set B contains the range.
Outputs: 9, 10, 12, 13, 15

4. A function $f: A \rightarrow B$ is bijective if, it is both injective and surjective.



$f \circ g$ is a function from $A \to C$, and by definition $f \circ g(x) := f(g(x))$.

$$f \circ g(1) = f(g(1)) = \beta$$

$$f \circ g(2) = f(g(2)) = \sigma$$

$$f \circ g(3) = f(g(3)) = \sigma$$

$$f \circ g(4) = f(g(4)) = \alpha$$

5. A function can be inverted if it is bijective.

4.2 Functions from real number to real number

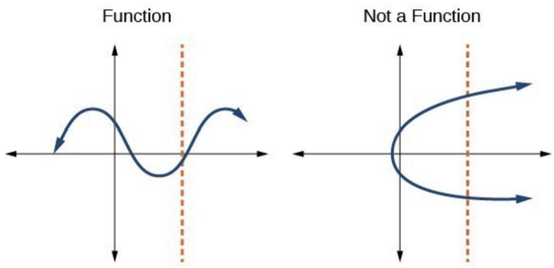
1. Domain and range

$f(x) = \sqrt{x+1}$. The expression is only meaningful when $x+1 \geqslant 0$, thus its domain is $[-1, +\infty)$

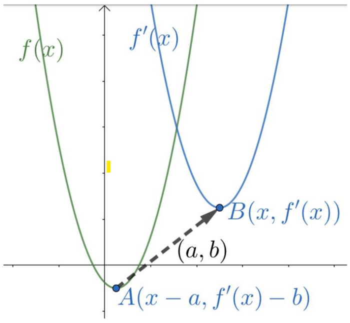
$f(x) = \sqrt{-2x-6}$. The expression is meaningful when $-2x-6 \geqslant 0$, its domain is $(-\infty, -3]$.

$f(x) = \dfrac{1+x}{x-2}$, $x-2=0$, its domain is $(-\infty, 2) \cup (2, \infty)$.

2. Graphs of Functions on the Coordinate Plane

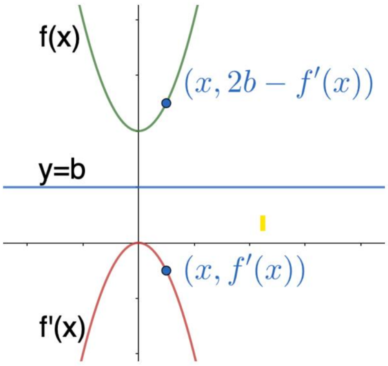


3. If we shift the graph of $y=f(x)$ to the right by a units, and up by b units, then we get the graph of a new function $f'(x)$. $f'(x) = f(x-a)+b$



4.If we reflect the graph of $y=f(x)$ across the horizontal line $y=b$, then we got

$f'(x)=2b-f(x)$



①Reflection Across a Vertical Line

If we reflect the graph of $y=f(x)$ across the vertical line $x=a$, then we get the graph of a new function $f'(x)$. Similar analysis gives $f'(x) = f(2a-x)$

②Reflection Across a Point

If we reflect the graph of $y=f(x)$ across the point (a,b), then we get the graph of a new function $f'(x)$. Similar analysis gives $f'(x) = 2b-f(2a-x)$

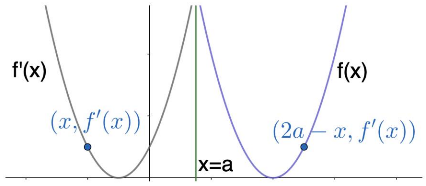
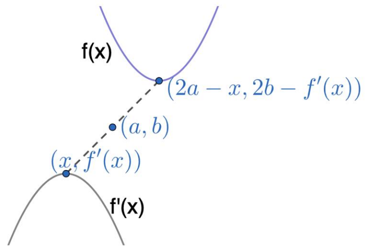


Figure 4.14: Reflection Across a vertical Line



$$
\begin{aligned}
f(-\frac{b}{a} - x) &= a(-b/a - x)^2 + b(-b/a - x) + c \\
&= ax^2 + 2bx + \frac{b^2}{a} - \frac{b^2}{a} - bx + c \\
&= ax^2 + bx + c = f(x)
\end{aligned}
$$

5. *Even and odd functions*

①*A function is even, if its domain is symmetric with respect to the origin, and $f(-x)=f(x)$ for all x in the domain.*

②*A function is odd, if its domain is symmetric with respect to the origin, and $f(-x)=-f(x)$ for all x in the domain.*

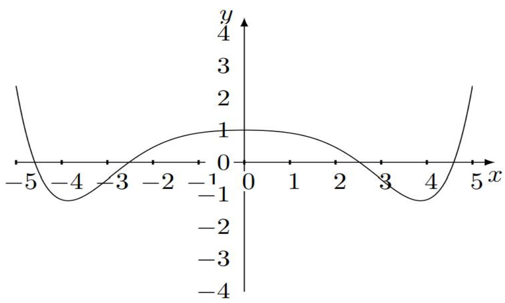


Figure 4.16: An even function

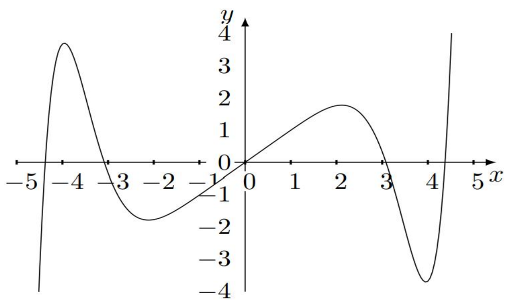


Figure 4.17: An odd function

③*odd+odd=odd*

even+even=even

odd - odd = even - even = even - odd = odd - even = not sure

6. Monotonicity

A function $f(x)$ is said to be monotonically increasing(decreasing) on a domain D, if $\forall x, y \in D, x - y > 0 \Rightarrow f(x) - f(y) \geqslant 0$ $(f(x) - f(y) \leqslant 0)$.

Discuss the monotonicity of $f(x) = -2x^2 + 8x - 7$.

① When $y < x \leqslant 2$, we have: $f(x) - f(y) = -2x^2 + 8x + 2y^2 - 8y = 2(x-y)(4-x-y) > 0$

The last step is due to $x > y$ and $x \leqslant 2$, $y < 2$. Thus $f(x)$ is increasing on $(-\infty, 2]$.

② Similarly, when $x > y \geqslant 2$, we have $f(x) - f(y) = -2x^2 + 8x + 2y^2 - 8y = 2(x-y)(4-x-y) < 0$

So $f(x)$ is decreasing on $[2, \infty)$.

Prove $f(x) = \sqrt{x}$ is an increasing function on $[0, +\infty)$.

For any $x > y \geqslant 0$, we can compute $f(x) - f(y) = \sqrt{x} - \sqrt{y} = \dfrac{x - y}{\sqrt{x} + \sqrt{y}} > 0$

So, $f(x) = \sqrt{x}$ is strictly increasing.

Exercises for Functions

1. Evaluate the piecewise defined function $f(x) = x^2\ (x<0)\ ;\ x+1\ (x\geqslant 0)$ at the indicated values:

(a) $f(-1) =$ ____

(b) $f(0) =$ ____

(c) $f(1) =$ ____

(a) $=1$, (b) $=1$, (c) $=2$.

2. Find the domain of the function

(a) $f(x) = \dfrac{\sqrt{x}}{2x^2+x-1}$

$x \in [0, \frac{1}{2}) \cup (\frac{1}{2}, +\infty)$

(b) $h(x) = \dfrac{x}{x^2-x-6}$

$x \neq 3$ and $x \neq 2$

(c) $f(x) = \sqrt{2x^2-3x-20} + \sqrt{x-3}$

$x \in [4, +\infty)$

3. For this question, find $f\circ g, g\circ f$ and their domains : $f(x) = \dfrac{1}{\sqrt{x}}, g(x) = x^2-4x$

$f\circ g = f(g(x)) = \dfrac{1}{\sqrt{x^2-4x}}$ $(x>4$ or $x<0)$

$g\circ f = g(f(x)) = \left(\dfrac{1}{\sqrt{x}}\right)^2 - \dfrac{4}{\sqrt{x}}$ $(x\neq 0)$

4. $f(x)$ **is an odd function in R, when** $x \leqslant 0, f(x)=2x^2-x$**, then** $f(1)=$ __

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5. Let $f(x)=x+4$**,** $g(x)=2x-5$**, find the following functions:**

(1) $g^{-1} \circ f^{-1}$

(2) $f^{-1} \circ g^{-1}$

(3) $(f \circ g)^{-1}$

(4) $(g \circ f)^{-1}$

(1) $\frac{x+1}{2}$ (2) $\frac{x-3}{2}$ (3) $\frac{x+1}{2}$ (4) $\frac{x-3}{2}$

6. Judgement function $f(x)=x^2-2x$ **'s Monotonicity in** $x \in [0,2]$

$f(x)=x^2-2x=(x-1)^2-1$

It's symmetry axis is $x=1$

When $x \in [0,1)$, $f(x)$ Monotone decreasing

When $x \in (1,2]$, $f(x)$ Monotone increasing