Grade 10 Calculus BC Study Guide

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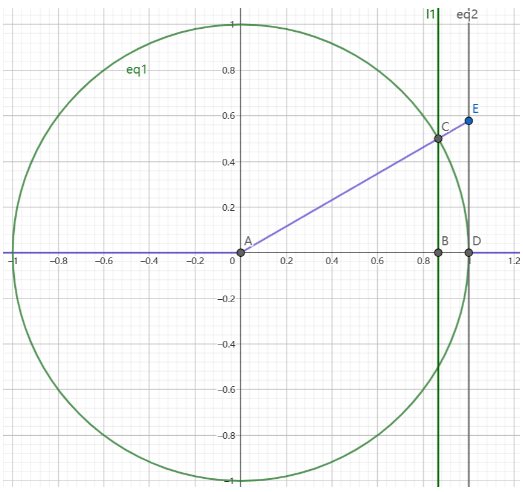
1. 摘要

本学习指导旨在梳理高一上学期期中考试前关于 calculus 课程的知识点，辅助同学对微积分建立系统化的认知，填补知识漏洞，以及为同学提供一些解题思路。本指导将主要介绍课程中没有提及的高中前置知识、课程讲解的关于极限与求导的知识、微积分题目的参考思路。此外，本指导还有拓展部分，帮助有兴趣学习数学的同学建立更深刻、更严格的认知。如果读者认为自己的基础足够，可以跳过高中前置知识部分。

2. 正文

2.1 三角函数

2.1.1 基本定义



图中令 AC 所在直线与 x 轴的辐角为 θ，则根据定义有：

1) $\sin\theta = \frac{BC}{1} = BC$

2) $\cos\theta = \frac{AB}{1} = AB$

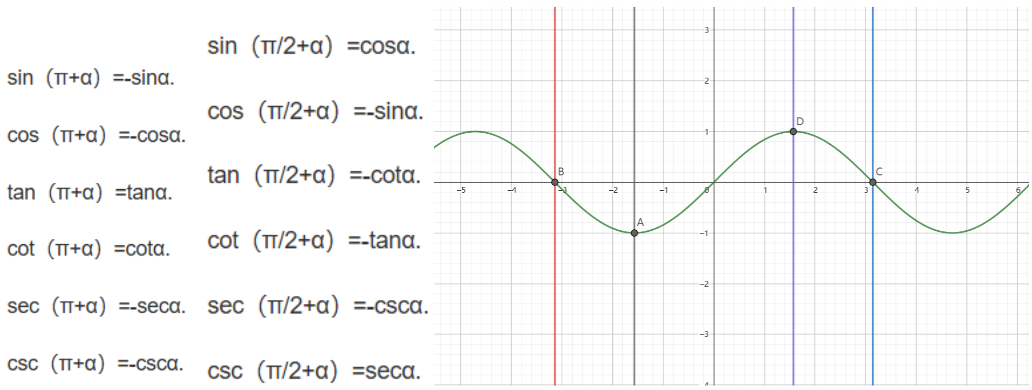
3) $\tan\theta = \frac{sin\theta}{cos\theta} = \frac{BC}{AB} = \frac{DE}{AD} = DE$

4) $\sec\theta = \frac{AC}{AB} = \frac{1}{cos\theta}$

5) $\csc\theta = \frac{AC}{BC} = \frac{1}{sin\theta}$

6) $\cot\theta = \frac{AB}{BC} = \frac{1}{tan\theta}$

2.1.2 诱导公式



诱导公式出自三角函数的周期性，右图用于辅助读者进行直觉化的理解。可以想象一点在原点与 D 间运动，则以其横坐标加 π 为横坐标的点在 C 右侧的与原点、点 C 间的函数图像相似的图像上，两点纵坐标显然互为相反数。读者可以尝试结合 cosx，tanx，secx 等函数图像自行验证诱导公式以加深印象(提示：中图中的公式需要结合函数图像的几何性质，左加右减)。以下关于三角恒等式的证明需要用到这些公式，建议读者仔细理解所有证明过程以获得更成体系的认知，这样可以避免死记硬背，即使忘了公式也能现推。

2.1.3 常用三角恒等式

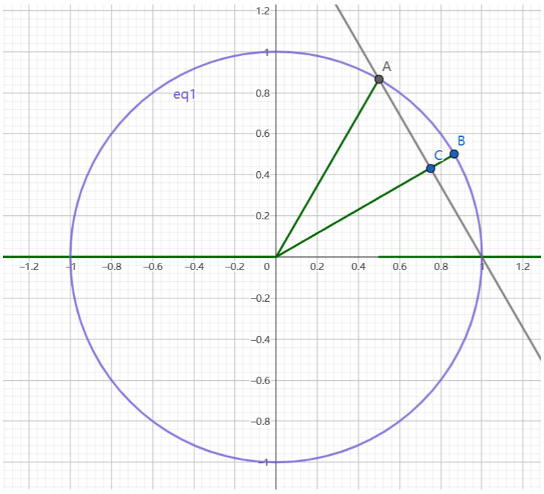
1) $cos(\alpha-\beta) = cos\alpha cos\beta + sin\alpha sin\beta$

$cos(\alpha+\beta) = cos\alpha cos\beta - sin\alpha sin\beta$

证明:

引理：$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|cos\theta = \vec{a}_x\vec{b}_x + \vec{a}_y\vec{b}_y$($\theta$为两向量间的夹角)，

此操作被称为求$\vec{a},\vec{b}$的点积，其几何意义为过$\vec{a}$向$\vec{b}$做垂线，后求垂足处模长与$\vec{b}$模长的乘积



由图，令 OA 的辐角为α, OB 的辐角为β，将 A, B 分别当作$\vec{a},\vec{b}$考虑，则

$$cos(\alpha-\beta)=OC=|\vec{a}||\vec{b}|cos(\alpha-\beta)=\vec{a}_x\vec{b}_x+\vec{a}_y\vec{b}_y=cos\alpha cos\beta+sin\alpha sin\beta$$

另一式可将β替换为-(- β)，再将$sin(-\beta)$替换为-sin(β)

2) $sin(\alpha+\beta)=sin\alpha cos\beta+sin\beta cos\alpha$;

$sin(\alpha-\beta)=sin\alpha cos\beta-sin\beta cos\alpha$

证明:

$$sin(\alpha+\beta)=-cos(\frac{\pi}{2}+\alpha+\beta)$$
$$=sin(\frac{\pi}{2}+\alpha)sin\beta-cos(\frac{\pi}{2}+\alpha)cos\beta$$
$$=sin\beta cos\alpha+sin\alpha cos\beta$$

另一式可将$sin(-\beta)$ *替换为* $-sin(\beta)$

3) $tan(\alpha-\beta)=\frac{tan\alpha-tan\beta}{1+tan\alpha cos\beta}$

$tan(\alpha-\beta)=\frac{tan\alpha+tan\beta}{1-tan\alpha cos\beta}$

下面看似跳跃较大的一步实则是令分数线上下同除$cos\alpha cos\beta$

$$tan(\alpha-\beta)=\frac{sin\alpha cos\beta-sin\beta cos\alpha}{cos\alpha cos\beta+sin\alpha sin\beta}=\frac{tan\alpha-tan\beta}{1+tan\alpha tan\beta}$$

同理可得另一式。

4) $sin2\theta=2sin\theta cos\theta$

$cos2\theta=cos^2\theta-sin^2\theta=1-2sin^2\theta=2cos^2\theta-1$

$tan2\theta=\frac{2tan\theta}{1-tan^2\theta}$

分别令之前的结论中的$\alpha=\beta=\theta$即可得到上面的结论，第二行的结论中用到了$cos^2\theta+sin^2\theta=1$

5) $1+tan^2\theta=sec^2\theta$

证明:

$$1+tan^2\theta=\frac{cos^2\theta+sin^2\theta}{cos^2\theta}=\frac{1}{cos^2\theta}=sec^2\theta$$

2.1.4 反三角函数

1) $arcsin(sinx)=x$

$arccos(cosx)=x$

$arctan(tanx)=x$

$arcsec(secx)=x$

$arccsc(cscx)=x$

$arccot(cotx)=x$

这些结论根据反三角函数的定义是显然的。

2) $arcsin(-x)=-arcsinx$

两侧取 sin 值即可证明该结论

3) c

根据$sinx=cos(\frac{\pi}{2}-x)$可得

4)

$$\begin{aligned}\arccos x &= \pi - \arccos(-x)\\ &= \frac{\pi}{2} - \arcsin x\\ &= \operatorname{arccot}\frac{x}{\sqrt{1-x^2}}\\ &= \arcsin\sqrt{1-x^2}\\ &= \arctan\frac{\sqrt{1-x^2}}{x}\end{aligned}$$

$$\begin{aligned}\operatorname{arccot} x &= \pi - \operatorname{arccot}(-x)\\ &= \frac{\pi}{2} - \arctan x\\ &= \arccos\frac{x}{\sqrt{1+x^2}}\\ &= \arcsin\frac{1}{\sqrt{1+x^2}}\\ &= \arctan\frac{1}{x}\end{aligned}$$

$$\begin{aligned}\arctan x &= -\arctan(-x)\\ &= \frac{\pi}{2} - \operatorname{arccot} x\\ &= \arcsin\frac{x}{\sqrt{1+x^2}}\\ &= \arccos\frac{1}{\sqrt{1+x^2}}\\ &= \operatorname{arccot}\frac{1}{x}\end{aligned}$$

由反三角函数几何意义易得。读者可通过画图自行验证。

5) 反三角函数的求导:

$$(arcsinx)` = \frac{1}{\sqrt{1-x^2}}$$
$$(arccosx)` = -\frac{1}{\sqrt{1-x^2}}$$
$$(arctanx)` = \frac{1}{1+x^2}$$

证明:

$$y = arcsinx$$

$$siny = x$$

$$y`cosy = 1$$

$$y` = \frac{1}{cosy} = \frac{1}{\sqrt{1-sin^2y}} = \frac{1}{\sqrt{1-x^2}}$$

同理可得其余两个结论

2.2 微积分基础知识

本模块基本完全由王明远同学撰写，与侯飞声无关，但如果读者对此模块知识有所疑问欢迎来找我们答疑。

Fundamental

Elementary Functions:

1 power x^n $x^0 = 1$

2 exponential n^x

3. logrethmic $\log_n x$

4. trigonometric

5. Inverse-trigonometric

6,

Chapter 1 Limit & continuity

1. limit of sequnce

2. Limit of a function

b One-sided Limit

$\lim_{x \to x_0} f(x)$ exist $\Longleftrightarrow \lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x)$

3. Limit involving infinities & asymptotes.

a Infinite limits

diverges 发散 f is unbounded

$\lim_{x \to a} f(x) = \pm\infty$

$x = a$ is a vertical asymptote

b. limits at infinity

If $\lim_{x \to \pm\infty} f(x) = \begin{cases} L_1 \\ L \end{cases}$

$y = L_1, L.$ is a horizontal asymptote

4. Techniques in finding the limit

a algebraic rules

b. indeterminate form

$\frac{f}{g}$, $\frac{0}{0}$, $\frac{\infty}{\infty}$, f^g, $0 \cdot \infty = \frac{0}{\frac{1}{\infty}} = \frac{\infty}{\frac{1}{0}}$

$\Downarrow$ $\frac{0}{0}$ $\Downarrow$ $\frac{\infty}{\infty}$

C. Squeeze / Sandwich Theorem

$\lim_{x \to a} f(x) = ?$

$\begin{cases} g \le f \le h \\ \lim g = \lim h = L \end{cases} \Rightarrow f = L$

$|g-L|$

$|f-L|$ $|h-L|$

d One special limit

$\lim_{x \to 0} \frac{\sin x}{x} = 1$

e Equivalent infinitesimal

$a \sim b$

$\Rightarrow a \to 0,\ b \to 0,\ \lim \frac{a}{b} = 1$

$x \sim \sin x \sim \tan x$

$x \sim \arcsin x \sim \arctan x$

$1 - \cos x \sim \frac{1}{2}x^2$

$\underline{1 - \frac{1}{2}x^2 \sim \cos x}$

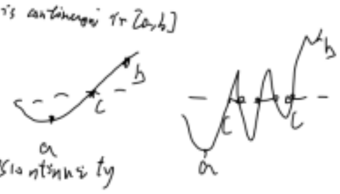
J. Continuity

a. Definition

$\lim_{x \to a} f(x) = f(a)$

b. Intermediate Value Theorem (IVT)

f is continuous in $[a, b]$



c. Discontinuity

- i. continuous : $\lim_{x \to a} f(x) = f(a)$
- ii. Removable : $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) \neq f(a)$
- iii Jump : $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$
- iv. Infinity : $\lim_{x \to a} f(x) = \pm\infty$
- V. oscillatory : $\lim_{x \to a}$ doesnt exist

essental

6. Another Special limit (NOT ∞)

Compound interest

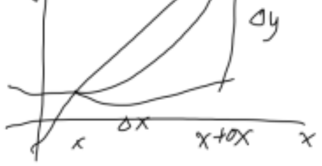
Chapter 2 Derivative

1. Average rate of change

a Defination $y = f(x)$

Average rate $= \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$

b. Geometric



slope of secant line = Arc

2. Derivative (instant

2. Derivative (instantaneous rate of change)

a. Definition

b. Geometric

c. Notation

i. Lagrange
$f', f'(a)$

ii Leibniz (Differential)
dx dy "infinitesimal"
ΔX

d. Differential

$\frac{dy}{dx} = f'(x)$, $dy = f'(x) \cdot dx$

e. Equation of Tangent line

$\text{slope} = f'(a) = \frac{y - f(a)}{x - a}$

$\therefore f'(a)x - f'(a)a = y - f(a)$

$\therefore y = f'(a)(x-a) + f(a)$

3. Differentiability

a. Def.
if f' exist at $x=a$
then, f is differentiable at $x=a$

b. Differentiable $\Rightarrow$ continuity

4. Derivative Function
$y = f(x)$

$\frac{dy}{dx}\Big|_{x=a} = f'(a)$

$\frac{dy}{dx} = f'(x)$

$$\frac{dy}{dx} = f'(x)$$

$$f^{(n)}(x)$$

$$\frac{d\left(\frac{df}{dx}\right)}{dx} = \frac{d^2 f}{dx^2}$$

5. Algebraic rule

(a) $(\text{constant})' = 0$

(b) $(kf)' = kf'$

$(\alpha f + \beta g)' = \alpha f' + \beta g'$

(c) $(f g)' = f'g + fg'$

(d) $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

$$\left(\frac{f}{g}\right)' = \frac{f'}{g} + f \cdot \left(-\frac{1}{g^2} \cdot g'\right)$$

$$= \frac{f' \cdot g - g' \cdot f}{g^2}$$

$$\left(\frac{1}{g}\right)' = -\frac{1}{g^2} \cdot g'$$

6. Derivative of

Fundamental Elementary Function

(a) Power x^a, base, a = const

$(x^a)' = a x^{a-1}$

(b) Exponential

a^x, base, $a > 0$

$(e^x)' = e^x$

$(a^x)' = a^x \ln a$

(c) Logarithmic

$(\ln x)' = \frac{1}{x}$

$(\log_a x)' = \frac{1}{x \ln a}$

(d) Trigonometric

$(\sin x)' = \cos x$

$(\cos x)' = -\sin x$

$(\tan x)' = \frac{1}{\cos^2 x} = \sec^2 x$

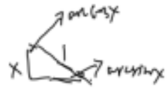
$$\frac{1}{\tan x} = (\cot x)' = -\tan x^{-2} \cdot \frac{1}{\cos^2 x} = -\csc^2 x$$

$$\frac{1}{\cos x} = (\sec x)' = \tan x \sec x$$

$$\frac{1}{\sin x} = (\csc x)' = -\csc x \cdot \cot x$$

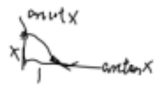
(e) Inverse Trigonometric

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$



$$\arccos(x) + \arcsin(x) = \frac{\pi}{2}$$

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$



$$(\arctan x)' = \frac{1}{1+x^2}$$

$$\operatorname{arccot} x + \arctan x = \frac{\pi}{2}$$

$$(\operatorname{arccot} x)' = -\frac{1}{1+x^2}$$

e.g. x^x

$$= e^{\ln x^x} = e^{x\ln x}$$

$$(x^x)' = e^{x\ln x} \cdot (\ln x + 1)$$

$$= x^x(\ln x + 1)$$

7. Chain Rule

$$y = f(g(x))$$

$$y' = \frac{dy}{dg(x)} \cdot \frac{dg(x)}{dx}$$

$$y' = f'(g(x)) \cdot g'(x)$$

8. Inverse Function Derivative

$$f f^{-1} = x$$

$$f'(f^{-1}) \cdot f^{-1\prime} = 1$$

$$f^{-1\prime} = \frac{1}{f' f^{-1}}$$

$$f^{-1\prime}(x) = \frac{1}{f'(f^{-1}(x))}$$

2.3 解题技巧与参考习题

为防止老师出本指导外题型之题增大考试难度，解题技巧将于日后统一在班

内讲解。此外，建议读者多多联系侯飞声或王明远，以针对读者需求或疑问提出针对性的建议。