. General function

Mapping & functions

Definition: one-one/ many- one

F: A —- B A is the domain of f B is the range of f

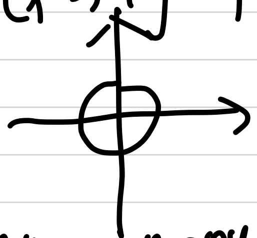
Every elements in A must be mapped to a elements in B

Injective单射 one to one

surjective 满射 every elements in the range has an error pointing to it

Bijective both injective and surjective

eg: $(x-2)^2+y^2=4$



one —many ✗

$y=\sqrt{16-x^2}$

设 $x=1$

$y=\sqrt{16-1}$

$y=\sqrt{15}$

只有一个y ✓

$x=-1$时 $y=\sqrt{15}$

many —one ✓

$|y|=4-x$

设 $x=1$

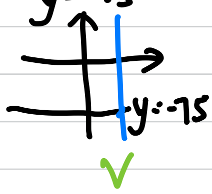
$|y|=4-1$

$|y|=3$

$y=\pm 3$

one—many ✗

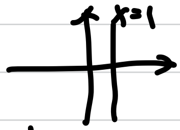
$y=-7.5$



✓

$x-1=0$

$x=1$



当$x=1$时 y任意

✗

Function composition

A—B—C = f(g(x))=f o g(x)

*Relation with set theory

How to compare the cardinality(sizes/ the numbers of elements in the sets) of two sets

finite count the numbers and then compare

Infinite for those who can established a one to one correspondence between their elements eg(the sets of all natural numbers &the sets of all integers' cardinality are the same) for those who can't their cardinality are not the same(eg the sets of all real numbers is bigger than N)

Find $f(g(h(x)))$ and $h(g(f(x)))$ with $f(x)=x^4$ $g(x)=x-5$ $h(x)=\sqrt{x}$ (Don't simplify)

$f(g(h(x)))$

将 $h(x)$ 代入 $g(x)$

$g(h(x))=\sqrt{x}-5$

将新求出的$g(x)$代入$f(x)$

$\therefore f(g(h(x)))=(\sqrt{x}-5)^4$

$h(g(f(x)))$

$\sqrt{x^4-5}$

Domain& range

Find the domain

elements under√square roots >=0

Elements as denominator can't be equal to 0

Domain

eg: $f(x) = \dfrac{1}{1+\dfrac{1}{1+\frac{1}{x}}}$

$\begin{cases} 1+\dfrac{1}{1+\frac{1}{x}} \neq 0 & x \neq -\frac{1}{2} \\ 1+\dfrac{1}{x} \neq 0 & x \neq -1 \\ x \neq 0 & x \neq 0 \end{cases}$

$(-\infty,-1) \cup (-1,-\frac{1}{2}) \cup (-\frac{1}{2},0) \cup (0,+\infty)$

$f(x) = \sqrt{2x^2-3x-20} + \sqrt{x-3}$

$2x^2-3x-20 \geq 0$ $\quad x-3 \geq 0$

$x \geq 3$

$\begin{matrix} 2 & -4 \\ 1 & 5 \end{matrix}$

$[3,+\infty)$

$(x-4)(2x+5) \geq 0$

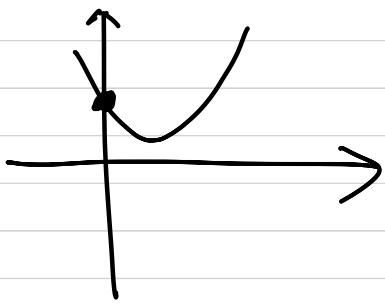
$-\frac{5}{2}$ $\quad 4$

$\therefore [4,+\infty)$

$(-\infty,-\frac{5}{2}] \cup [4,+\infty)$

★ Given the domain of $f(x) = \dfrac{\sqrt[3]{x-5}}{kx^2+4kx+3}$ is R, then what is the range of k?

记 $kx^2+4k+3 \neq 0$!!!



∵(恒)过 (0,3) 且不能与x轴有交点

$\therefore k>0 \quad \Delta<0$

$\therefore k \quad (0,\frac{3}{4})$

Given the Domain of $f(x)$ is $[0,1]$ then what is the Domain of $f(\sqrt{x}-2)$

Domain 即 x 范围. $0 \leq x \leq 1$ 但. $f(x)$ 的 x 和 $f(\sqrt{x}-2)$ 中的 x

此x非彼x

唯一相同就是 ( ) 中范围不变.

$0 \leq (\ ) \leq 1$

$0 \leq \sqrt{x}-2 \leq 1$

$2 \leq \sqrt{x} \leq 3$

$4 \leq x \leq 9$

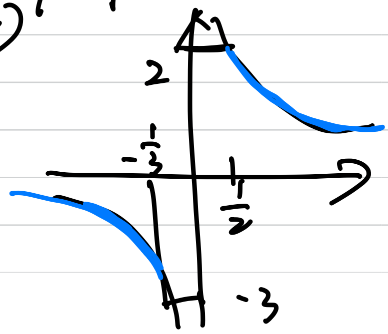
Given the Domain of $f(x+1)$ is $[-2,3)$ then find the Domain of $f(\frac{1}{x}+2)$

$-2 \leq x \leq 3$

$-1 \leq x+1 \leq 4$

$-1 \leq \frac{1}{x}+2 \leq 4$

$-3 \leq \frac{1}{x} \leq 2$



$(-\infty,-\frac{1}{3}] \cup [\frac{1}{2},+\infty)$

Range:

2种求法

1. 配平

2. 变成 $\square + \frac{\square}{\square}$ 形式

Find the range of $f(x)=\sqrt{5-4x-x^2}$　　即为 $y=\sqrt{5-4x-x^2}$ y范围

★ 将 $5-4x-x^2$ 变形 $\longrightarrow -(x+2)^2+9$

$-(x+2)^2+9 \geq 0$

可得　$9 \geq -(x+2)^2+9 \geq 0$

$3 \geq \sqrt{-(x+2)^2+9} \geq 0$

$\therefore [0,3]$

Find the range of $f(x)=\frac{-4x+3}{2x-1}$. Can you generalized to the range of $f(x)=\frac{ax+b}{cx+d}$ ($c\,ac \neq 0$)

$$\begin{array}{r} -2 \\ 2x-1 \overline{\smash{\big)}\,-4x+3} \\ -4x+2 \\ \hline 1 \end{array}$$

$-2+\frac{1}{2x-1}$

只要 $x \neq \frac{1}{2}$ 就可

$\{y \mid y \neq -2\}$

$f(x)=\frac{ax+b}{cx+d}$

$$\begin{array}{r} \frac{a}{c} \\ cx+d \overline{\smash{\big)}\,ax+b} \\ ax+\frac{ad}{c} \\ \hline b-\frac{ad}{c} \end{array}$$

$\frac{a}{c}+\frac{b-\frac{ad}{c}}{cx+d}$

$\frac{a}{c}+\frac{\frac{bc-ad}{c}}{cx+d}$

$\because cx+d \neq 0$

$c \neq 0$

if $bc=ad$　$\{y \mid y=\frac{c}{a}\}$

if $bc \neq ad$　$\{y \mid y \neq \frac{c}{a}\}$

Find the range of $f(x)=\frac{1-x^2}{1+x^2}$

① 将 $\frac{1-x^2}{1+x^2}=-1+\frac{2}{1+x^2}$　　$1+x^2 \geq 1$

$$\begin{array}{r} -1 \\ 1+x^2 \overline{\smash{\big)}\,1-x^2} \\ -1-x^2 \\ \hline 2 \end{array}$$

无限大　$-1+\frac{2}{1}=-1+2=1$

无限小　趋近于 -1

$(-1,1]$

Graphs and functions under the coordinate system

Shifting& translation 即上加下减左加右减 but why?

A units to the right

X’=x +a ——x=x’-a

Y’= y———y=y’

Y=f(x)—y’=f(x’-a)—-将x’y’用xy表示 y=x-a (此 x非彼x)

A units above

X’=x———-x=x’

Y’=y+a ——y=y’-a

y=f(x)—y’-a=f(x’)——y’=f(x’)+a---将x’y’用x y表示 (此 x非彼x 所以y=x+a

绝对值?!

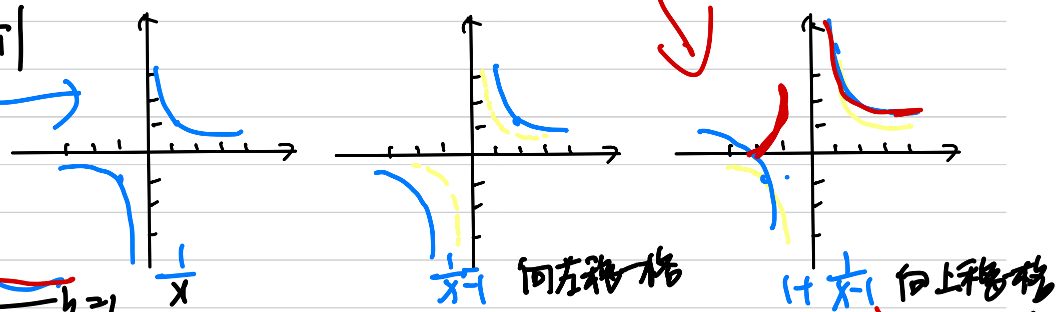
y<0 向上翻折

eg: $y=\left|\frac{x}{x-1}\right|$ sketch the following Graph

$\left|\frac{x}{x-1}\right| = \left|1+\frac{1}{x-1}\right|$

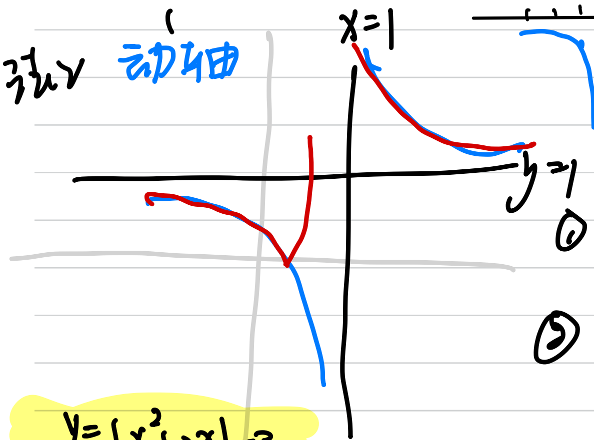
动函数

动轴



$\frac{1}{x}$ 　 $\frac{1}{x-1}$ 向右移1格 　 $1+\frac{1}{x-1}$ 向上移1格

即可以看作上加下减 左加右减



x=1

y=1

① $x-1\neq0$, $x\neq1 \to x\neq1$

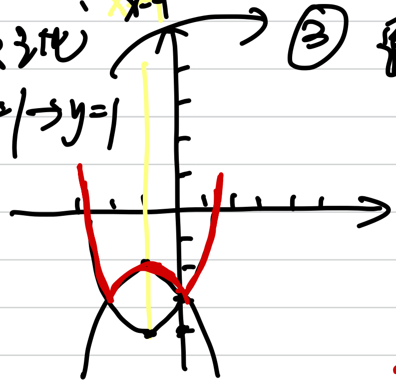
② 找最高次项系数比 $\frac{x}{x} = 1 \to y=1$

③ 用试代点判断 象限.

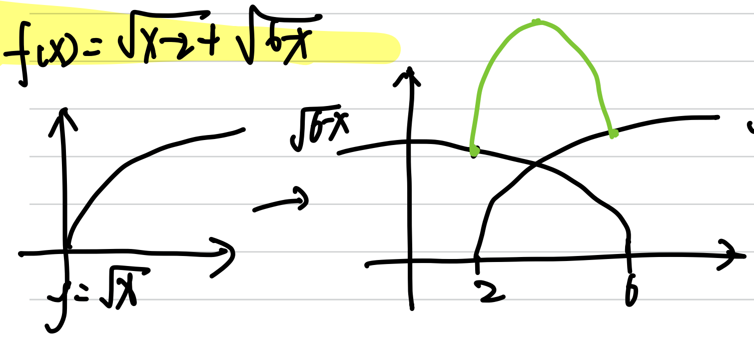
$y=|x^2+2x|-3$

$x^2+2x>0$ 　 $y=x^2+2x-3$ 　 $x\ (-\infty,-2)\cup(0,+\infty)$

$x^2+2x<0$ 　 $y=-x^2-2x-3$ 　 $x\ (-2,0)$



$f(x)=\sqrt{x-2}+\sqrt{6-x}$



$y=\sqrt{x}$ 　 $\sqrt{6-x}$ 　 $\sqrt{x-2}$

2 　 6

合集 domain range

2张graph=

Given a function $f(x) = x^4 - 2x^3 + 1$ (Don't simplify)

(1) The graph of function $g(x)$ is obtained by shifting the graph of $f(x)$ 2 units to the right and 1 units down, find $g(x)$

根据上加下减左加右减

$$f(x) = (x-2)^4 - 2(x-2)^3 + 1 - 1$$ 不化简

(2) The graph of function $h(x)$ is obtained by reflecting the graph of $f(x)$ with respect to the line $x = -2$ find $h(x)$

$\because \frac{x_1 + x_2}{2} =$ 对称轴

$\therefore \frac{x_{new} + x}{2} = -2$

$x_{new} = -4 - x$ 代入

$\rightarrow f(x) = (-4-x)^4 - 2(-4-x)^3 + 1$

(3) The graph of function is $s(x)$ is obtained by reflecting the graph of $f(x)$ with respect to the line $y = 2$. find $s(x)$

$\frac{y_1 + y_2}{2} = 2$

$y_{new} = 4 - y$

$f(x) = 4 - (x^4 - 2x^3 + 1)$

$f(x) = 4 - x^4 + 2x^3 - 1$

(4) The graph of function $T(x)$ is obtained by reflecting the graph of $f(x)$ with respect to the point $(-1, -1)$ find $T(x)$

$$\begin{cases} \frac{x_1 + x_2}{2} = -1 \\ \frac{y_1 + y_2}{2} = -1 \end{cases}$$

$f(x) = -2 - [(-2-x)^4 + (-2-x)^3 + 1]$

Even & odd functions

Even : $f(x)=f(-x)$

Odd: $f(x)=-f(-x)$

$f(0)=0$

$f(x)+f(-x)=0$

Odd*odd=even   odd+odd=odd   odd*even=odd   odd+even=nor

eg: $f(x)$ is an odd function in $\mathbb{R}$, when $x\le 0$ $f(x)=2x^2-x$, then $f(1)=$

法1 $\because f(x)$ is an odd function

$\therefore f(x)=-f(-x)$

$x>0$

$f(-x)=2x^2+x$

$f(x)=-f(-x)=-2x^2-x$

将1代入

法2 $f(-1)=2\times 1+1$

$f(-1)=3$

$-f(1)=3$

$f(1)=-3$

$f(x)$ is an even function a, $g(x)$ is an odd function, then

A) $f(x)+|g(x)|$ is an even function ✓

$f(-x)+|g(-x)|$ 将$-x$代入看是否与$f(x)+|g(x)|$关系

if 不变 even

if 相反数 odd

$\because f(x)=f(-x)$ $g(-x)=-g(x)$

$\therefore f(x)+|-g(x)|$

$=f(x)+|g(x)|$ even

B) $|f(x)|+g(x)$ is an even function

$|f(-x)|+g(-x)$

$=|f(x)|-g(x)$ neither

For an odd function $f(x)$, $x\in \mathbb{R}$ $f(2-\sqrt{3})+f(\frac{1}{2+\sqrt{3}})=$

① 去分母

$f(2-\sqrt{3})+f(\frac{1}{2+\sqrt{3}}\times\frac{2-\sqrt{3}}{2-\sqrt{3}})=f(2-\sqrt{3})+f(\frac{2-\sqrt{3}}{2^2-3})=f(2-\sqrt{3})+f(-2+\sqrt{3})$

$\because f(x)$ is an odd function

$f(x)=-f(-x)$

$\therefore f(2-\sqrt{3})+f(-2+\sqrt{3})=0$

State whether each function is even or odd prove your statement

a) $f(x)=|x+1|-|x-1|$

$f(-x)=|-x+1|-|-x-1|$

$=|-x+1|-|x+1|$

$=-|x+1|+|-x+1|$

$=-|x+1|+|-(x-1)|$

负号可以删掉

相反数

odd function

b) $f(x)=(x-1)\sqrt{\frac{1+x}{1-x}}$

法1 $f(-x)=(-x-1)\sqrt{\frac{1-x}{1+x}}$

$f(-x)=-(x+1)\sqrt{\frac{1-x}{1+x}}$

neither

法2 求 Domain

if Domain 都不对称

它就不可能是奇/偶

$\frac{1+x}{1-x}\ge 0$ $1-x\ne 0$

$1\ne x$

$(1+x)(1-x)\ge 0$ $[-1,1)$

$(x+1)(x-1)\le 0$

Inverse function 反函数

F-1(x) is the inverse function of f(x) 值得注意的是! Many-one 没有反函数!!!

函数图像关于y=x对称

eg: $f(x)=\sqrt{2+5x}$

$y=\sqrt{2+5x}$

$y^2=2+5x$

$y^2-2=5x$

$\frac{y^2-2}{5}=x$ 将x,y换位置、

$y=\frac{x^2-2}{5}$

$f(x)=\sqrt{9-x^2}$ $(0\le x\le 3)$

$y=\sqrt{9-x^2}$

$y^2=9-x^2$

$x^2=9-y^2$

$x=\sqrt{9-y^2}$

$y=\sqrt{9-x^2}$ 同时别忘求x范围 $(0\le x\le 3)$

$f(x)=x^2+x$ $(x\ge -\frac{1}{2})$

$y=x^2+x$

配方!!

$y=x^2+x+\frac{1}{4}-\frac{1}{4}$

$y=(x+\frac{1}{2})^2-\frac{1}{4}$

$y+\frac{1}{4}=(x+\frac{1}{2})^2$

$\sqrt{y+\frac{1}{4}}=x+\frac{1}{2}$

$\sqrt{y+\frac{1}{4}}-\frac{1}{2}=x$

$y=\sqrt{x+\frac{1}{4}}-\frac{1}{2}$ $(x\ge -\frac{1}{4})$

∵为反函数 ∴ $x\ge -\frac{1}{2}$ 成为反函数中y的range $y\ge -\frac{1}{2}$ 则 $x\ge -\frac{1}{4}$

$f(x)=-|x-1|-2$ $x\ge 1$

$y=-|x-1|-2$

$y=-(x-1)-2$

$y=-x+1-2$

$y=-x-1$

$x=-1-y$

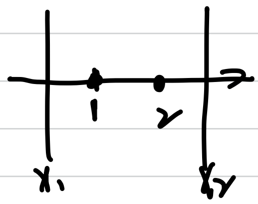
$y=-1-x$ $(x\le -2)$

$f(x)=\begin{cases}-x & x\le 0\\ x^2-2x & x>0\end{cases}$ 不单调. ∴f(x)无反函数

The sufficient and necessary condition for the proposition "$f(x)=x^2-2ax-3$ is invertible on [1,2] is

即理解为 函数"...."在[1,2]有反函数,

有反函数条件 → 在此范围内单调



$x=-\frac{b}{2a}=-\frac{-2a}{2\times 1}=a$ $a\le 1$ / $a\ge 2$

$(-\infty,1]$ & $[2,+\infty)$

Given $f(3x+1)=4x+7$, then what is $f(x)$? 整体代换

$3x+1=t$

$3x=t-1$

$x=\frac{t-1}{3}$

$f(t)=\frac{t-1}{3}\times 4+7$

$f(t)=\frac{4(t-1)}{3}+7$

$f(t)=\frac{4(t-1)}{3}+\frac{21}{3}$

$f(t)=\frac{4t-4+21}{3}$

$f(t)=\frac{4t+17}{3}$ 用x代换t

$f(x)=\frac{4x+17}{3}$

直接考反函数性质

1) 则 f(x) Inverse function

2) f(x) is symmetrical with g(x) with line y=x

3) f(x) & g(x) are mutually inverse

Monotonicity 单调性

increasing 增函数
$x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

decreasing 减函数
$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

复合函数 同增异减

$f(g(x))$

$x_1 < x_2$ $f\uparrow$ $g\uparrow$
$g(x_1) < g(x_2)$
$f(g(x_1)) < f(g(x_2))$

$x_1 < x_2$ $f\downarrow$ $g\downarrow$ 同
$g(x_1) > g(x_2)$
$f(g(x_1)) < f(g(x_2))$
increasing function

$x_1 < x_2$ $f\uparrow$ $g\downarrow$
$g(x_1) > g(x_2)$
$f(g(x_1)) < f(g(x_2))$

$x_1 < x_2$ $f\downarrow$ $g\uparrow$ 异
$g(x_1) < g(x_2)$
$f(g(x_1)) > f(g(x_2))$
decreasing function

eg: For an decreasing function $f(x)$, let $m = f(a^2)$, $n = f(a-1)$ then $m \leq n$

$f(a^2) - f(a-1) = a^2 - (a-1) = a^2 - a + 1 = (a - \frac{1}{2})^2 + \frac{3}{4} > 0$
$f(a^2) > f(a-1)$
$\because$ Decreasing.
$\therefore$ $m < n$

prove that $f(x) = \sqrt{x^2+1} - x$ is decreasing on $(0, +\infty)$ by definition.

注意定义域范围 eg: Discuss the monotonicity of $f(x) = \frac{2x+4}{x-2}$
$\boxed{分类,\ x > 2\ \&\ x < 2}$
Decreasing Decreasing
$\therefore$ Decreasing

Let $x_1 < x_2$ 假设 $x_1 < x_2$ if $y_1 < y_2$ increase if $y_1 > y_2$ decrease

$f(x_1) - f(x_2) = \sqrt{x_1^2+1} - x_1 - (\sqrt{x_2^2+1} - x_2)$
$= \sqrt{x_1^2+1} - x_1 - \sqrt{x_2^2+1} + x_2$
$= \sqrt{x_1^2+1} - \sqrt{x_2^2+1} - x_1 + x_2$
分子有理化!!!
$= \frac{x_1^2+1 - x_2^2 - 1}{\sqrt{x_1^2+1} + \sqrt{x_2^2+1}} - (x_1 - x_2)$
$= (x_1 - x_2)\left(\frac{x_1 + x_2}{\sqrt{x_1^2+1} + \sqrt{x_2^2+1}} - 1\right) \Rightarrow$ 负

负 <1

$\sqrt{x_1^2} + \sqrt{x_2^2} < \sqrt{x_1^2+1} + \sqrt{x_2^2+1}$

$\therefore$ $f(x_1) - f(x_2) > 0$
$f(x_1) > f(x_2)$

$\therefore$ Decreasing..

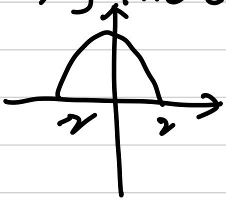
根据上面的两题我们可以发现 Monotonicity & $f(x_1)$, $f(x_2)$关系 & x_1, x_2关系

知2求1

The even function $f(x)$ for $x \in [-2,2]$ is decreasing on $[0,2]$. Then what is the range of m such that $f(1-m) < f(m)$

$\because$ even function

$\therefore$ $[-2,0]$ increasing



$|1-m| > |m|$

$(1-m)^2 > m^2$

$1-2m+m^2 > m^2$

$1 > 2m$

$\frac{1}{2} > m$

$-2 \le 1-m \le 2$

$-2 \le m \le 2$

$\Downarrow$

$-1 \le m \le 3$

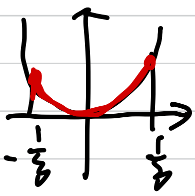
$-2 \le m \le 2$

$\Downarrow$

$-1 \le m \le 2$

$\therefore$ $[-1, \frac{1}{2})$

For an even function $f(x)$ increasing on $[0,+\infty)$, solve the inequality $f(2x-1) < f(\frac{1}{3})$



$-\frac{1}{3} < 2x-1 < \frac{1}{3}$

$2x-1 < \frac{1}{3}$

$2x < \frac{1}{3}+\frac{3}{3}$

$2x < \frac{4}{3}$

$x < \frac{2}{3}$

$-\frac{1}{3} < 2x-1$

$-\frac{1}{3}+1 < 2x$

$-\frac{1}{3}+\frac{3}{3} < 2x$

$\frac{2}{3} < 2x$

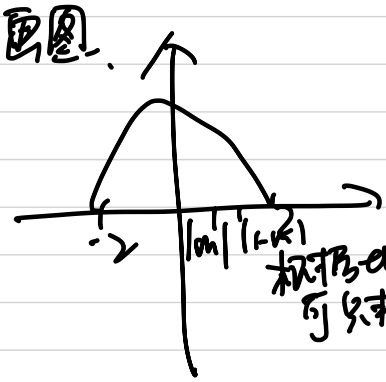
$\frac{2}{3} \times \frac{1}{2} < x$

$\frac{1}{3} < x$

$(\frac{1}{3}, \frac{2}{3})$

① 画图

②



根据even对称性

可只看右半边

$\Downarrow$

根据f(x)关系可以

写出绝对值 $|m| < |1-m|$

平方两边

$(m)^2 < (1-m)^2$

$\cancel{m^2} < 1-2m+\cancel{m^2}$

$2m < 1$

$m < \frac{1}{2}$